The hyperfinite algorithm

for “sequences” of knots

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Overview

- Limits of sequences of knots - Hyperfinite knots

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- Knots - Invariants - Quandles - The CJKLS invariant

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- The CJKLS invariant in the thermodynamic limit: the free energy per crossing
- Hyperfinite knots: examples
- What happens when another CJKLS invariant is chosen?

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- This talk is devoted to showing how this can be done plus ...

Given a sequence what happens if we change topologies?

- ... better ask this question again after the first question is answered ...

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- The induced $f^{\sim}$ embeds this quotient space in M
- We can then regard $\mathcal{K}_f$ as a metric subspace of M
- We take the closure of $\mathcal{K}_f$ in the topology of M and call it $\overline{\mathcal{K}_f}$

A picture:

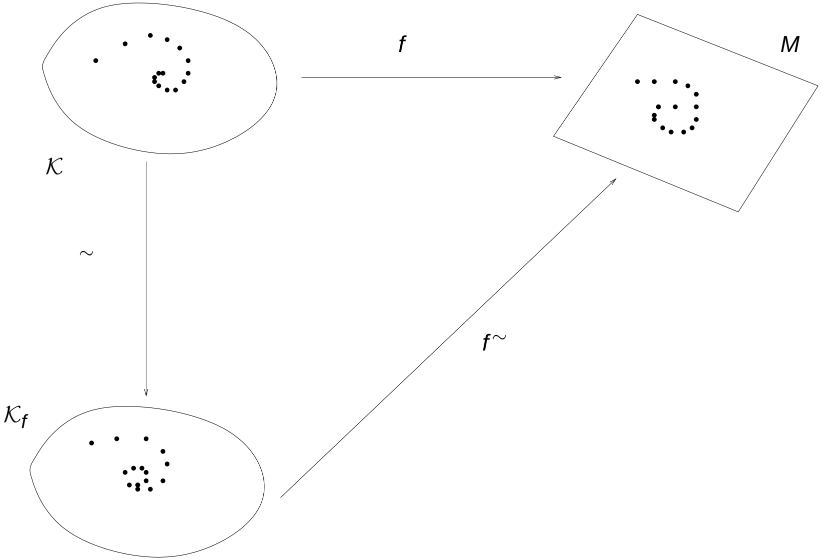

Figure: The “hyperfinite” algorithm

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  - ϕ - 2-co-cycle in $H^2(X;A)$, i.e.,
    1. $\phi(a,a) = 1$
    2. $\phi(a,b)\phi(a*b,c) = \phi(a,c)\phi(a*c,b*c)$

    Example: $\phi \equiv 1_A$

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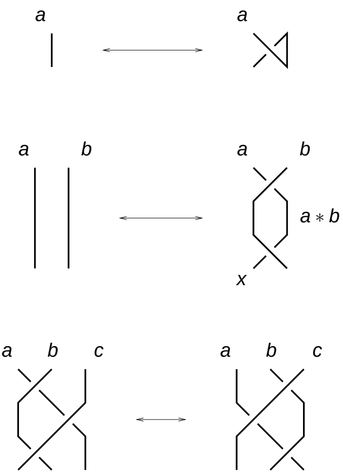
- ... it’s the quandle ...

A crash course on quandle theory (cont'd)

REIDEMEISTER MOVES



QUANDLE AXIOMS

I	$a * a = a$
II	$x * b = a$
III	$(a * b) * c = (a * c) * (b * c)$

Figure: Quandle Axioms vs. Reidemeister moves

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- Count colorings instead (homomorphisms to a fixed quandle)
- or use the CJKLS invariant
  - which is a sum over these colorings
  - and specializes to the number of colorings when using the trivial co-cycle

Assembling the CJKLS invariant...

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  - $A = \mathbb{Z}_2 \cong (\, t \mid t^2 = 1\,)$
  - $$\phi(a, b) = \begin{cases} 1\,, & \text{if } a = b \text{ or } a = T \text{ or } b = T \\ t\,, & \text{otherwise} \end{cases}$$

The CJKLS invariant of the trefoil:

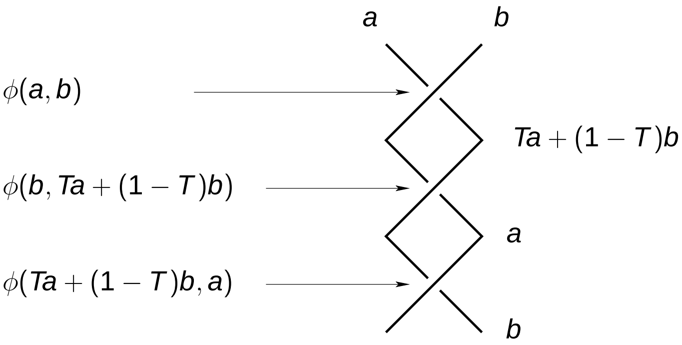

Figure: The colorings and evaluation of the 2-cocycle at crossings for the trefoil

The CJKLS invariant of the trefoil (cont'd):

- Set

$$\Phi(a,b) := \phi(a,b)\cdot\phi(b, Ta+(1-T)b)\cdot\phi(Ta+(1-T)b, a) =$$
$$= \begin{cases} t\,, & \text{if } a \neq b \\ 1\,, & \text{if } a = b \end{cases}$$

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- and

$$Z(\text{Trefoil}) = \sum_{a,b\in\{0,1,T,1+T\}} t^{\bar{\delta}_{a,b}} \quad = \quad 4(1+3t) \quad \longleftrightarrow \quad (4,12)$$

The CJKLS invariant of K_2:

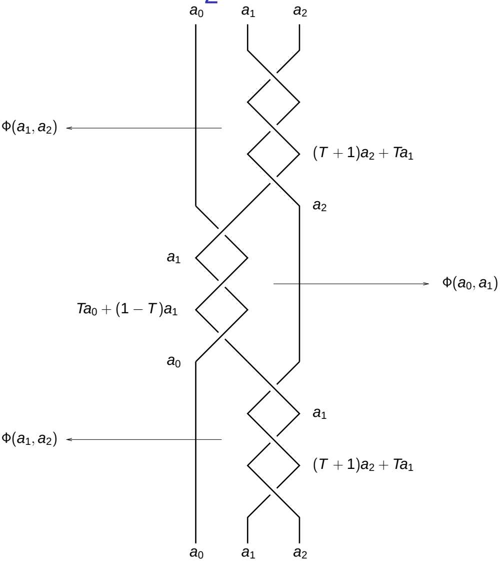

Figure: K_2, upon closure of the braid, endowed with a coloring by S_4

The CJKLS invariant of K_2 (cont'd):

- $$Z(K_2) = \sum_{a_0,a_1,a_2 \in \{0,1,T,1+T\}} \Phi(a_1,a_2)\Phi(a_0,a_1)\Phi(a_1,a_2) =$$
$$= \sum_{a_0,a_1,a_2 \in \{0,1,T,1+T\}} t^{\bar{\delta}_{a_0,a_1}} = 4^2(1+3t)$$
$$\longleftrightarrow \quad (4^2, 4^2 \cdot 3)$$

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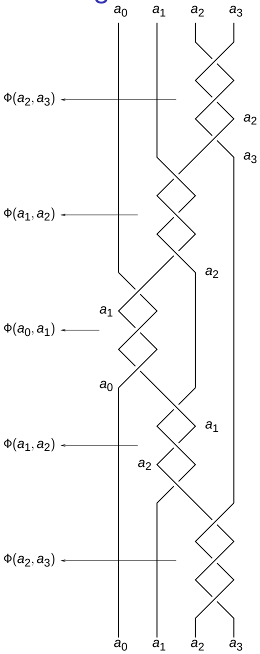

The CJKLS invariant of K_3:

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$$= \sum_{a_0,\ldots,a_3 \in \{0,1,T,1+T\}} t^{\bar{\delta}_{a_0,a_1}} = 4^3(1+3t)$$

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The CJKLS invariant of K_n:

- $$Z(K_n) = 4^n(1+3t) \longleftrightarrow (4^n, 4^n \cdot 3)$$

The sequence of CJKLS invariants of the free energy per crossing, f, for K_n:

-

$$Z(K_1) = (4, 4 \cdot 3)$$

$$f(K_1) = \left(\frac{\ln(4)}{3}, \frac{\ln(4 \cdot 3)}{3}\right) = \left(\frac{2\ln(2)}{3}, \frac{2\ln(2) + \ln(3)}{3}\right)$$

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- $$Z(K_n) = (4^n, 4^n \cdot 3)$$

  $$f(K_n) = \left(\frac{\ln(4^n)}{6n-3}, \frac{\ln(4^n 3)}{6n-3}\right) = \left(\frac{2n\ln(2)}{6n-3}, \frac{2n\ln(2) + \ln(3)}{6n-3}\right)$$

  $$\xrightarrow[n \to \infty]{} \left(\frac{\ln(2)}{3}, \frac{\ln(2)}{3}\right)$$

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- cf. P.L., Sequences of Knots and Their Limits,
  in Geometry and Physics: XVI International Fall Workshop,
  R. L. Fernandes et al (eds.),
  AIP Conference Proceedings, **1023**, 183-186, 2008

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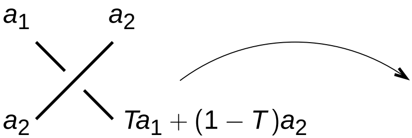
- ... by ideals of the sort $(p, L(T))$, for prime p and Laurent poly L, ...
- and $a * b = Ta + (1 - T)b$, in the indicated quotient.
- Example: $X = S_4 \cong \mathbb{Z}_2[T, T^{-1}]/(T^2 + T + 1)$
  $a * b := Ta + (1 - T)b$ ...

... relevant evidence? - Proof of Theorem (cont'd)

- The Burau representation of the braid group and its connections with colorings by Alexander quandles:

$$\sigma_i \longmapsto I_{i-1} \oplus \begin{pmatrix} 0 & T \\ 1 & 1-T \end{pmatrix} \oplus I_{N}$$



$$(a_1 \quad a_2)\begin{pmatrix} 0 & T \\ 1 & 1-T \end{pmatrix}$$

Figure: The Burau representation of the braid group and its connections with colorings by Alexander quandles

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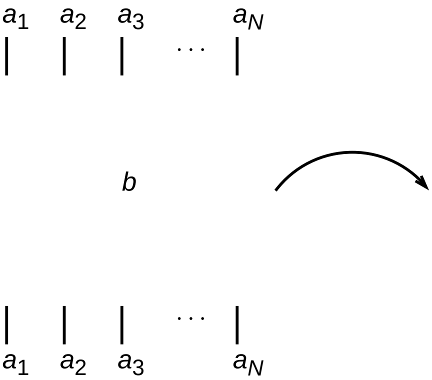

$$(a_1\ a_2\ a_3\ \cdots\ a_N)B(d) =$$

$$= (a_1\ a_2\ a_3\ \cdots\ a_N)$$

Figure: The coloring equation for the knot represented by the closure of the braid b, whose Burau matrix is $B(d)$. The equalities are to be understood in the quotient corresponding to the Alexander quandle at stake.

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- The Burau matrices are invertible hence form a finite group, hence, for each of them, there is a finite order.

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- Let M be a positive integer such that $[B(b)]^M = Id$.

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- Let M be a positive integer such that $[B(b)]^M = Id$.
- Let $|A|$ be the order of A, an abelian group. Let X denote the Alexander quandle at stake and choose a 2-co-cycle ϕ.

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- For each positive integer n, write

$$n = M|A|l_n + r_n,$$

where $l_{,} r_n$ are positive integers, and $0 \leq r_n < m|A|$.

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- Then,

$$Z(K_n) =$$

$$= \sum_{\substack{a_1,\ldots,a_N \in X \\ \text{s.t. ...}}} \prod_{\tau \in c(b^n)} \phi^{\epsilon_\tau} = \sum_{\substack{a_1,\ldots,a_N \in X \\ \text{s.t. ...}}} \left(\left(\prod_{\tau \in c(b^M)} \phi^{\epsilon_\tau} \right)^{|A|} \right)^{l_n} \cdot \prod_{\tau \in c(b^{r_n})}$$

$$= \sum_{\substack{a_1,\ldots,a_N \in X \\ \text{s.t. ...}}} \Big(Id_A \Big)^{l_n} \cdot \prod_{\tau \in c(b^{r_n})} \phi^{\epsilon_\tau} = \sum_{\substack{a_1,\ldots,a_N \in X \\ \text{s.t. ...}}} \prod_{\tau \in c(b^{r_n})} \phi^{\epsilon_\tau}$$

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- Again

$$Z(K_n) = \underbrace{\sum_{\substack{a_1,\ldots,a_N \in X \\ \text{s.t. } \ldots}}}_{\text{at most } M \text{ systems of coloring equations}} \underbrace{\prod_{\tau \in c(b^{r_n})}}_{\text{at most } |A| \text{ values of } A} \phi^{\epsilon_\tau}$$

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- If C is the maximum number of solutions over all M systems of equations, ...
- then there are at most $MC|A|$ distinct values for $Z(K_n)$ i.e., this sequence is bounded.

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- If C is the maximum number of solutions over all M systems of equations, ...
- then there are at most $MC|A|$ distinct values for $Z(K_n)$ i.e., this sequence is bounded.
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- Further assuming that the crossing number of this sequence is increasing then
- $$f(K_n) = \left(\frac{Z_1(K_n)}{c(K_n)}, \ldots, \frac{Z_{|A|}(K_n)}{c(K_n)} \right) \underbrace{\longrightarrow}_{n \mapsto \infty} (\underbrace{0, \ldots, 0}_{|A| \text{ entries}})$$

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  - The sequence represents a hyperfinite knot in any “(Alexander) formalism” – stability
  - This hyperfinite knot has the “same” invariant in each “(Alexander) formalism” – “sharpness”

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- What if we now choose a different formalism on the same sequence?

Some calculations...(cont'd)

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- Then

$$0 \leq \lim_{n\to\infty} f^{i}_{X',A',\phi'}(K_n) \leq \frac{1}{c_{K_n}} \ln\left(|X|^{c_{K_n}} \cdot |X|^{u_{K_n}} \right) =$$

$$= \frac{1}{c_{K_n}} (c_{K_n} + u_{K_n}) \ln(|X|)$$

$$\underset{n\to\infty}{\longrightarrow} (1 + l) \ln(|X|)$$

Some calculations...(cont'd)

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- Upshot:
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- If a sequence converges wrt one CJKLS-formalism then it has converging subsequences on **any** other formalism
- Let us call this "quasi-stability" of hyperfinite knots wrt the CJKLS invariants' topologies

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  Hyperfinite knots via the CJKLS invariant in the thermodynamic limit,

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  Sequences of Knots and Their Limits,

  in Geometry and Physics: XVI International Fall Workshop,
  R. L. Fernandes et al (eds.),
  AIP Conference Proceedings, **1023**, 183-186, 2008