Asymptotic spectral analysis of Toeplitz operators on symplectic manifolds

Yuri A. Kordyukov

Ufa Federal Research Centre RAS and Novosibirsk State University

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The setting

X a (compact) smooth manifold;
(L, h^L, ∇^L) a Hermitian line bundle on X:

- $L \to X$ a complex line bundle on X:
  locally, over some open $\Omega \subset X$,
  $L|_\Omega \cong \Omega \times \mathbb{C}$; $C^\infty(\Omega, L|_\Omega) \cong C^\infty(\Omega)$;
- h^L a Hermitian structure in the fibers of L:

$$s, s' \in L \to (s, s')_{h^L} \in \mathbb{C},$$

- ∇^L a connection (covariant derivative): for $U \in C^\infty(X, TX)$

$$\nabla^L_U : C^\infty(X, L) \to C^\infty(X, L),$$

  which is Hermitian:

$$\nabla^L_U (s, s')_{h^L} = (\nabla^L_U s, s')_{h^L} + (s, \nabla^L_U s')_{h^L}, \quad s, s' \in C^\infty(X, L).$$

Example

- $X = \mathbb{R}^d$.
- $L = X \times \mathbb{C} \to X$ the trivial line bundle, $C^\infty(X, L) \cong C^\infty(X)$.
- The Hermitian structure is given by $h \in C^\infty(X)$: for $Z \in \mathbb{R}^d$

$$|s|_h^2 = h(Z)|s|^2, \quad s \in L_Z = \{Z\} \times \mathbb{C};$$

- The connection: for any $U \in C^\infty(X, TX)$,
  $\nabla_U^L : C^\infty(X, L) \to C^\infty(X, L)$ is the first order differential operator:

$$\nabla_U^L = \frac{\partial}{\partial U} + \Gamma(U),$$

  $\Gamma = \sum_{j=1}^d \Gamma_j(Z) dZ^j \in \Omega^1(X)$ is the connection one-form;
- ∇^L is Hermitian $\Leftrightarrow \Gamma + \bar{\Gamma} = -h^{-1} dh$.

The Bochner-Laplacian

Let (L, h^L, ∇^L) be a Hermitian line bundle on X.
The connection can be considered as an operator

$$\nabla^L = d + \Gamma : C^\infty(X, L) \to C^\infty(X, T^*X \otimes L)$$

Fiix a Riemannian metric g on X.
We have L^2-inner products on $C^\infty(X, L)$ and $C^\infty(X, T^*X \otimes L)$:

$$(s, s')_{L^2(X,L)} = \int_X (s(x), s'(x))_{h^L} dv_g(x), \quad s, s' \in C^\infty(X, L).$$

The formally adjoint operator

$$(\nabla^L)^* : C^\infty(X, T^*X \otimes L) \to C^\infty(X, L).$$

For $s \in C^\infty(X, L), s' \in C^\infty(X, T^*X \otimes L)$:

$$(\nabla^L s, s')_{L^2(X, T^*X \otimes L)} = (s, (\nabla^L)^* s')_{L^2(X,L)}.$$

The Bochner-Laplacian

Definition

The Bochner-Laplacian Δ^L associated with a Hermitian line bundle (L, h^L, ∇^L):

$$\Delta^L = (\nabla^L)^* \nabla^L : C^\infty(X, L) \to C^\infty(X, L).$$

If $\{e_j\}_{j=1,\dots,d}$ is a local orthonormal frame of TX, then

$$\Delta^L = -\sum_{j=1}^{d} \left[(\nabla^L_{e_j})^2 - \nabla^L_{\nabla^{TX}_{e_j} e_j} \right],$$

where ∇^{TX} the Levi-Civita connection of g.

Example: magnetic Laplacian

- $X = \mathbb{R}^d$.
- $L = X \times \mathbb{C}$ the trivial line bundle.
- The Hermitian structure ($h(Z) = 1$):

$$|s(Z)|_h^2 = |s(Z)|^2.$$

- The connection form

$$\Gamma = -i\mathbf{A}, \quad \mathbf{A} = \sum_{j=1}^{d} A_j(Z)dZ_j$$

  is a real-valued one form (a magnetic potential).
- The Riemannian metric is the standard metric $g = \sum_{j=1}^{d} dZ_j^2$.
- The Bochner-Laplacian is the magnetic Schrödinger operator:

$$\Delta^L = -\sum_{j=1}^{d} \left(\frac{\partial}{\partial Z_j} - iA_j(Z) \right)^2.$$

The curvature

Let (L, h^L, ∇^L) be a Hermitian line bundle on X.
The curvature of ∇^L is the differential two-form R^L on X:

$$R^L(U, V) = \nabla^L_U \nabla^L_V - \nabla^L_V \nabla^L_U - \nabla^L_{[U,V]}, \quad U, V \in TX.$$

For the connection $\nabla^L_U = \frac{\partial}{\partial U} + \Gamma(U)$, its curvature is given by

$$R^L = d\Gamma.$$

For the magnetic Laplacian $\Delta^L = -\sum_{j=1}^d \left(\frac{\partial}{\partial Z_j} - iA_j(Z)\right)^2$,

$$R^L = -i\mathbf{B}$$

where $\mathbf{B} = d\mathbf{A}$ is a real-valued two form (the magnetic field):

$$\mathbf{B} = \sum_{j,k=1}^d B_{jk}(Z) dZ_j \wedge dZ_k, \quad B_{jk} = \frac{\partial A_k}{\partial Z_j} - \frac{\partial A_j}{\partial Z_k}.$$

Non-degeneracy assumption

We will assume that the differential two form

$$\omega = \frac{i}{2\pi} R^L$$

is nondegenerate (of full rank):

$$(\omega(U,V) = 0 \text{ for any } V \in TX) \Rightarrow U = 0.$$

For the magnetic Laplacian $\Delta^L = -\sum_{j=1}^{d} \left(\frac{\partial}{\partial Z_j} - iA_j(Z) \right)^2$,

$$\omega = \frac{1}{2\pi} \mathbf{B}$$

is non-degenerate.
In particular, the dimension d is even: $d = 2n$.

Relation with geometric quantization

- (X, ω) a symplectic manifold, $\dim X = 2n$, so it is a classical phase space.
- A Hermitian line bundle (L, h^L, ∇^L) on X, satisfying:

$$\frac{i}{2\pi} R^L = \omega$$

  is called a prequantum line bundle.
- (X, ω) is called quantizable $\Leftrightarrow$ there exists a prequantum bundle ($\Leftrightarrow [\omega] \in H^2(X, \mathbb{Z})$).
- In geometric quantization scheme (Kostant-Souriau), the operators act on sections of L (prequantization).

Example: the 2-sphere

- X the two-dimensional sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

  equipped with the Riemannian metric induced by the standard Euclidean metric in $\mathbb{R}^3$.
- ω is a scalar multiple of the volume 2-form dx_g:

$$\omega = s\, dx_g, \quad s \in \mathbb{R} .$$

- (X, ω) is quantizable $\Leftrightarrow$ the area $4\pi s = n \in \mathbb{Z}$.
- The corresponding prequantum line bundle (L_n, ∇_n) is a well-known Wu-Yang magnetic monopole, which provides a natural topological interpretation of Dirac's monopole of magnetic charge $g = nh/2e$.

The renormalized Bochner-Laplacian

- $J_0 : TX \to TX$ a skew-adjoint linear endomorphism:

$$\omega(u,v) = g(J_0 u, v), \quad u, v \in TX;$$

- τ is a smooth function on X given by

$$\tau(x) = \pi \operatorname{Tr}[(-J_0^2(x))^{1/2}], \quad x \in X.$$

- L^p the p-th tensor power of L, $p \in \mathbb{N}$;
- $\nabla_U^{L^p} : C^\infty(X, L^p) \to C^\infty(X, L^p)$ the induced connection on L^p:

$$\nabla_U^{L^p} = \frac{\partial}{\partial U} + p\Gamma^L(U), \quad U \in TX.$$

Definition (V. Guillemin - A. Uribe, 1988)

The renormalized Bochner-Laplacian Δ_p acts on $C^\infty(X, L^p)$:

$$\Delta_p = \Delta^{L^p} - p\tau.$$

Magnetic Laplacian

- $X = \mathbb{R}^{2n}$, $L = X \times \mathbb{C}$ the trivial line bundle.
- The Hermitian structure $|s(Z)|_h^2 = |s(Z)|^2$.
- The connection form $\Gamma = -i\mathbf{A}$, where $\mathbf{A} = \sum_{j=1}^{2n} A_j(Z)dZ_j$ is a real-valued one form.
- The Bochner-Laplacian

$$\Delta^{L^p} = -\sum_{j=1}^{2n}\left(\frac{\partial}{\partial Z_j} - ipA_j(Z)\right)^2, \quad p = \frac{1}{\hbar}.$$

- $J_0 = \frac{1}{2\pi}B$, where $B : TX \to TX$ be a skew-adjoint operator

$$\mathbf{B}(u,v) = g(Bu,v), \quad u,v \in TX.$$

- $\tau(Z) = \frac{1}{2}\,\mathrm{Tr}(B^*B)^{1/2} = \mathrm{Tr}^+(B)$.

Complex manifolds

- $X = \mathbb{C}^n$, $L = X \times \mathbb{C}$ the trivial line bundle.
- The Hermitian structure is given by $h \in C^\infty(X)$: for $z = x + iy \in \mathbb{C}^n$
$$|s|_h^2 = h(z)|s|^2, \quad s \in L_z;$$
- The Hermitian connection
$$\nabla^L = d + \Gamma, \quad \Gamma + \bar{\Gamma} = -h^{-1}dh;$$

  Assume that Γ is compatible with the complex structure of $\mathbb{C}^n$ (a holomorphic Hermitian connection — the Chern connection), then, Γ is a $(1,0)$-form:
$$\Gamma = \partial \log h = \sum_{j=1}^{n} h^{-1} \frac{\partial h}{\partial z_j} dz_j;$$

Complex manifolds

- The curvature $R = d\Gamma$ is a purely imaginary 2-form: $(1,1)$-form

$$R = \bar{\partial}\partial \log h.$$

- For the symplectic form ω, we have

$$\omega = \frac{i}{2\pi}\bar{\partial}\partial \log h.$$

- ω is positive if $h = e^{-\varphi}$, $\varphi : X \to \mathbb{C}$ a smooth strictly plurisubharmonic function:

$$\omega = \frac{i}{2\pi}\sum_{j,k=1}^{n} \frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k, \quad \left(\frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k}\right)_{j,k=1,\ldots,n} > 0.$$

Kähler manifolds

- A particular case: the Hermitian structure is given by

$$|s|_h^2 = h(z)|s|^2, \quad h(z) = e^{-\frac{\pi}{2}|z|^2};$$

- The connection form

$$\Gamma = \partial \log h = -\pi \sum_{j=1}^{n} \bar{z}_j dz_j;$$

- The symplectic form ω is the canonical symplectic form:

$$\omega = \frac{i}{2\pi}\bar{\partial}\partial \log h = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

- J_0 is a complex structure, the standard complex structure on $\mathbb{C}^n$.
- ω is a Kähler form on $\mathbb{C}^n$ and $(\mathbb{C}^n, J_0)$ is a Kähler manifold.

- $\tau(z) = \pi \operatorname{Tr}[(-J_0^2(z))^{1/2}] = 2\pi n, z \in X.$
- The Kodaira-Laplacian:

$$\square^{L^p} = \bar{\partial}^{L^p *} \bar{\partial}^{L^p} = -\frac{1}{2} \sum_{j=1}^{n} \left(\frac{\partial}{\partial z_j} - \pi p \bar{z}_j \right) \frac{\partial}{\partial \bar{z}_j}.$$

- For the renormalized Bochner-Laplacian, we have

$$\Delta_p = 2\square^{L^p}.$$

$$\begin{aligned} \Delta_p &= -\sum_{j=1}^{n} \left[(\nabla^{L^p}_{\partial/\partial x_j})^2 + (\nabla^{L^p}_{\partial/\partial y_j})^2 \right] - 2\pi np \\ &= -\sum_{j=1}^{n} \left[\left(\frac{\partial}{\partial x_j} - \pi p \bar{z}_j \right)^2 + \left(\frac{\partial}{\partial y_j} - \pi i p \bar{z}_j \right)^2 \right] - 2\pi np \\ &= -\sum_{j=1}^{n} \left[\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} - \pi p \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right]. \end{aligned}$$

The almost complex structure

- $J_0 : TX \to TX$ a skew-adjoint linear endomorphism such that

$$\omega(u,v) = g(J_0 u, v), \quad u, v \in TX;$$

- $J : TX \to TX$ the linear endomorphism given by

$$J = J_0(-J_0^2)^{-1/2}.$$

- J is an almost complex structure on X, $J^2 = -Id_{TX}$, compatible with ω and g:

$$\omega(Ju, Jv) = \omega(u,v), \quad g(Ju, Jv) = g(u,v), \quad u, v \in TX.$$

- ω is positive: for $u \in TX \setminus 0$

$$\omega(u, Ju) = -g(JJ_0 u, u) = g((-J_0^2)^{1/2} u, u) > 0.$$

- If $J_0 = J$ and J is integrable, then (X, J) is a Kähler manifold.

Spectral gap property

Theorem (Guillemin-Uribe, 1988; Ma-Marinescu, 2002)

There exists $C_L > 0$ such that for any p

$$\sigma(\Delta_p) \subset [-C_L, C_L] \cup [2p\mu_0 - C_L, +\infty),$$

where the constant μ_0 is given by

$$\mu_0 = \inf_{u \in T_xX, x \in X} \frac{iR^L_x(u, J(x)u)}{|u|^2_g}.$$

Example

$$\mu_0 = \inf_{u \in TX} \frac{|(B^*B)^{1/4}u|^2_g}{|u|^2_g} = \inf_{x \in X} \inf (B^*B(x))^{1/2}.$$

Generalized Bergman projection

- $\mathcal{H}_p$ the linear subspace of $L^2(X, L^p)$ spanned by the eigensections of Δ_p corresponding to eigenvalues in $[-C_L, C_L]$ (small eigenvalues).
- $P_{\mathcal{H}_p}$ the orthogonal projection in $L^2(X, L^p)$ onto $\mathcal{H}_p$ (generalized Bergman projection).

Example

(X, ω) a compact Kähler manifold, L a holomorphic line bundle:

- $\Delta_p = 2\Box^{L^p}$, where $\Box^{L^p}$ is the Kodaira Laplacian on L^p:

$$\sigma(\Delta_p) \subset \{0\} \cup [2p\mu_0 - C_L, +\infty),$$

- $\mathcal{H}_p$ is the space $H^0(X, L^p)$ of holomorphic sections of L^p.
- $P_{\mathcal{H}_p}$ the usual Bergman projection.

Toeplitz operators

A Toeplitz operator is a sequence of bounded linear operators $T_p : L^2(X, L^p) \to L^2(X, L^p), p \in \mathbb{N}$:

- For any $p \in \mathbb{N}$, we have

$$T_p = P_{\mathcal{H}_p} T_p P_{\mathcal{H}_p}.$$

- There exists a sequence $g_l \in C^\infty(X)$ such that

$$T_p = P_{\mathcal{H}_p} \left(\sum_{l=0}^{\infty} p^{-l} g_l \right) P_{\mathcal{H}_p} + \mathcal{O}(p^{-\infty}),$$

  i.e. for any natural k there exists $C_k > 0$ such that

$$\left\| T_p - P_{\mathcal{H}_p} \left(\sum_{l=0}^{k} p^{-l} g_l \right) P_{\mathcal{H}_p} \right\| \leqslant C_k p^{-k-1}.$$

Toeplitz operators

For any p, the operator T_p acts on a finite-dimensional space $\mathcal{H}_p$. The dimension of $\mathcal{H}_p$ is given, for p large enough, by the Riemann-Roch-Hirzebruch formula

$$d_p := \dim \mathcal{H}_p = \langle \mathrm{ch}(L^p)\, \mathrm{Td}(TX), [X] \rangle.$$

Here $\mathrm{ch}(L^p)$ is the Chern character of L^p and $\mathrm{Td}(TX)$ is the Todd class of the tangent bundle TX considered as a complex vector bundle with complex structure J. In particular,

$$d_p \sim p^n \int_X \frac{\omega^n}{n!}, \quad p \to \infty.$$

Algebra of Toeplitz operators

Theorem (Yu.K., 2017, Ioos-Lu-Ma-Marinescu, 2017)

The product $T_{f,p}T_{g,p}$ of the Toeplitz operators

$$T_{f,p} = P_{\mathcal{H}_p} f P_{\mathcal{H}_p}, \quad T_{g,p} = P_{\mathcal{H}_p} g P_{\mathcal{H}_p}, \quad f, g \in C^\infty(X),$$

is a Toeplitz operator. It admits the asymptotic expansion

$$T_{f,p}T_{g,p} = \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),p} + \mathcal{O}(p^{-\infty}),$$

with some $C_r(f,g) \in C^\infty(X)$, where C_r are bidifferential operators:

$$C_0(f,g) = fg, \quad C_1(f,g) - C_1(f,g) = i\{f,g\},$$

where $\{f,g\}$ is the Poisson bracket on $(X, 2\pi\omega)$.

Wells and localization of eigenfunctions

A self-adjoint Toeplitz operator T_p with principal symbol h:

$$T_p = P_{\mathcal{H}_p}\left(\sum_{l=0}^{\infty} p^{-l} g_l\right) P_{\mathcal{H}_p} + \mathcal{O}(p^{-\infty}), \quad g_0 = h.$$

An obvious lower bound:

$$(T_p v, v) \geqslant (\inf_{x\in X} h(x) + \mathcal{O}(p^{-1}))\|v\|, \quad v \in L^2(X, L^p).$$

Indeed, for $T_{h,p} = P_{\mathcal{H}_p} h P_{\mathcal{H}_p}$ and $v \in \mathcal{H}_p$,

$$(T_{h,p} u, u) = \int_X h(x)|u(x)|^2 dv_g(x) \geqslant \inf_{x\in X} h(x)\|u\|^2.$$

Wells and localization of eigenfunctions

Let $h_0 \in \mathbb{R}$. Suppose λ_p is a sequence of eigenvalues of T_p such that

$$\lambda_p \leqslant h_1 < h_0, \quad p \in \mathbb{N},$$

then the corresponding normalized eigensection u_p of T_p:

$$T_{h,p}u_p = \lambda_p u_p, Pu_p = u_p, \|u_p\| = 1$$

should be localized in the well

$$U_{h_0} = \{x \in X : h(x) \leqslant h_0\}.$$

Classically forbidden domain

$$X \setminus U_{h_0} = \{x \in X : h(x) > h_0\}.$$

Tunneling estimates

Assume that there exist $C > 0$ and $a > 0$ such that for any $p \in \mathbb{N}$ and $(x, y) \in X \times X$,

$$|(T_p - P_{\mathcal{H}_p} h P_{\mathcal{H}_p})(x, y)| < C p^{-1} e^{-a\sqrt{p} d(x,y)}.$$

Theorem (Y.K., 2018)

Suppose that $u_p \in \mathcal{H}_p$, $\|u_p\| = 1$, is a sequence of eigenfunctions of T_p, $T_p u_p = \lambda_p u_p$, such that

$$\lambda_p \leqslant h_1 < h_0, \quad p \in \mathbb{N}.$$

There exist $\alpha > 0$ and $C_1 > 0$ such that, for any $p \in \mathbb{N}$,

$$\int_X e^{2\alpha\sqrt{p} d(x, U_{h_0})} |u_p(x)|^2 dv_g(x) < C_1.$$

A self-adjoint Toeplitz operator T_p with principal symbol h:

$$T_p = P_{\mathcal{H}_p}\left(\sum_{l=0}^{\infty} p^{-l} g_l\right) P_{\mathcal{H}_p} + \mathcal{O}(p^{-\infty}), \quad g_0 = h.$$

Without loss of generality, we will assume that the principal symbol h satisfies the condition:

$$\min_{x\in X} h(x) = 0.$$

The spectrum of T_p consists of a finite number of eigenvalues

$$\lambda_p^0 \leqslant \lambda_p^1 \leqslant \ldots \leqslant \lambda_p^{d_p-1}.$$

The asymptotic properties of λ_p^m in the semiclassical limit $p \to \infty$.

The Bergman projection

Let $x_0 \in X$ be a non-degenerate minimum of h:

$$\operatorname{Hess} h(x_0) > 0.$$

The model operator for Δ_p at x_0:

$$\mathcal{L}_0 = -\sum_{j=1}^{2n}\left(\frac{\partial}{\partial e_j} + \frac{1}{2}R^L_{x_0}(Z, e_j)\right)^2 - \tau(x_0),$$

- $\{e_j\}_{j=1,\dots,2n}$ is an orthonormal base in $T_{x_0}X$.
- $\frac{\partial}{\partial U}$ the ordinary differentiation operator on $T_{x_0}X$ in the direction $U \in T_{x_0}X$.

$\mathcal{P}_{x_0}$ the orthogonal projection in $L^2(T_{x_0}X)$ to the kernel of $\mathcal{L}_0$ (the Bergman projection of $\mathcal{L}_0$).

The Bergman kernel

$\mathcal{J}_{x_0} : T_{x_0}X \to T_{x_0}X$ is a skew-adjoint operator such that

$$R^L(u,v) = g(\mathcal{J}u, v), \quad u, v \in TX.$$

Actually, $\mathcal{J} = -2\pi i J_0$.
We choose an orthonormal basis $\{e_j : j = 1, \dots, 2n\}$ of $T_{x_0}X$:

$$\mathcal{J}_{x_0} e_{2k-1} = a_k e_{2k}, \quad \mathcal{J}_{x_0} e_{2k} = -a_k e_{2k-1}, \quad a_k > 0 \quad k = 1, \dots, n.$$

We use this basis to define the coordinates Z on $T_{x_0}X \cong \mathbb{R}^{2n}$ as well as the complex coordinates z on $\mathbb{C}^n \cong \mathbb{R}^{2n}$, $z_j = Z_{2j-1} + iZ_{2j}, j = 1, \dots, n$.
The smooth Schwartz kernel (with respect to $dv_{TX}(Z)$) of $\mathcal{P}_{x_0}$:

$$\mathcal{P}(Z, Z') = \frac{1}{(2\pi)^n} \prod_{j=1}^{n} a_j \exp\left(-\frac{1}{4}\sum_j a_j(|z_j|^2 + |z_j'|^2 - 2z_j\bar{z}_j')\right).$$

The Toeplitz operator $\mathcal{T}_{x_0}$ in $L^2(T_{x_0}X)$ (the model operator for T_p at x_0):

$$\mathcal{T}_{x_0} = \mathcal{P}_{x_0}(q_{x_0}(Z) + g_1(x_0))\mathcal{P}_{x_0},$$

- $q_{x_0}(Z)$ the second order term in the Taylor expansion of h at x_0 (in normal coordinates near x_0):

$$q_{x_0}(Z) = \left(\frac{1}{2}\mathrm{Hess}\, h(x_0)Z, Z\right), \quad Z \in T_{x_0}X,$$

  ia positive quadratic form on $T_{x_0}X \cong \mathbb{R}^{2n}$.
- g_1 is the second coefficient in the asymptotic expansion for $\{T_p\}$.

It is an unbounded self-adjoint operator in $L^2(T_{x_0}X)$ with discrete spectrum. The eigenvalues of $\mathcal{T}_{x_0}$ do not depend on the choice of normal coordinates, and the lowest eigenvalue is simple.

Eigenvalue asymptotic expansions

Assume that the principal symbol h satisfies the following conditions:

- $h(x) \geqslant 0$ for any $x \in X$;
- $\min_{x\in X} h(x) = 0$;
- Each minimum is non-degenerate.

Then $U_0 = \{x \in X : h(x) = 0\}$ is a finite set (discrete wells):

$$U_0 = \{x_1, \dots, x_N\}.$$

Let $\mathcal{T}$ be the self-adjoint operator on $L^2(T_{x_1}X) \oplus \dots \oplus L^2(T_{x_N}X)$

$$\mathcal{T} = \mathcal{T}_{x_1} \oplus \dots \oplus \mathcal{T}_{x_N}.$$

Theorem (Y.K. (2018), A. Deleporte (2017, for Kähler manifolds))

Let $\{\lambda_p^m\}$ be the increasing sequence of the eigenvalues of T_p on $\mathcal{H}_p$ (counted with multiplicities) and $\{\mu_m\}$ the increasing sequence of the eigenvalues of $\mathcal{T}$ (counted with multiplicities). Then, for any fixed m, λ_p^m has an asymptotic expansion, when $p \to \infty$, of the form

$$\lambda_p^m = p^{-1}\mu_m + p^{-3/2}\phi_m + \mathcal{O}(p^{-2}).$$

Theorem (Y.K. (2018), A. Deleporte (2017, for Kähler manifolds))

If μ is a simple eigenvalue of $\mathcal{T}_{x_j}$ for some j, then there exists a sequence λ_p of eigenvalues of T_p on $\mathcal{H}_p$ which has a complete asymptotic expansion of the form

$$\lambda_p^0 \sim p^{-1}\sum_{k=0}^{+\infty} a_k p^{-k}, \quad a_0 = \mu.$$

Toeplitz operator $\mathcal{T}(Q)$ in $L^2(\mathbb{C}^n)$

$$\mathcal{T}(Q) = \mathcal{P}Q : \ker \mathcal{L}_0 \subset L^2(\mathbb{C}^n) \to \ker \mathcal{L}_0 \subset L^2(\mathbb{C}^n),$$

where $Q = Q(z, \bar{z})$ is a polynomial in $\mathbb{C}^n$ and $\mathcal{P}$ is the orthogonal projection in $L^2(\mathbb{C}^n)$ to the kernel of $\mathcal{L}_0$.

Fock space $\mathcal{F}_n$ is the space of holomorphic functions F in $\mathbb{C}^n$ such that $e^{-\frac{1}{2}|z|^2} F \in L^2(\mathbb{C}^n)$.

$\mathcal{F}_n$ is a closed subspace in $L^2(\mathbb{C}^n; e^{-\frac{1}{2}|z|^2} dz)$.

The orthogonal projection $\Pi : L^2(\mathbb{C}^n; e^{-\frac{1}{2}|z|^2} dz) \to \mathcal{F}_n$:

$$\Pi F(z) = \frac{1}{\pi^n} \int_{\mathbb{C}^n} \exp\left(-|z'|^2 + z \cdot \bar{z}'\right) F(z')\, dz'\, d\bar{z}'.$$

Consider the isometry $S : L^2(\mathbb{C}^n; e^{-\frac{1}{2}|z|^2}dz) \to L^2(\mathbb{C}^n)$ given, for $u \in L^2(\mathbb{C}^n; e^{-\frac{1}{2}|z|^2}dz)$, by

$$Su(z) = \frac{\prod_{j=1}^n a_j}{2^n} e^{-\frac{1}{4}\sum_j a_j|z_j|^2} u\left(\phi(z)\right),$$

where $\phi : \mathbb{C}^n \to \mathbb{C}^n$ is a linear isomorphism given by

$$\phi(z) = \left(\frac{\sqrt{a_1}}{\sqrt{2}} z_1, \ldots, \frac{\sqrt{a_n}}{\sqrt{2}} z_n\right), \quad z \in \mathbb{C}^n.$$

It is easy to see that $S\Pi S^{-1} = \mathcal{P}$. It follows that $S(\mathcal{F}_n) = \ker \mathcal{L}_0$ and

$$\mathcal{T}(Q) = S\mathcal{T}^0(Q \circ \phi^{-1})S^{-1}.$$

where $\mathcal{T}^0(Q)$ is a Toeplitz operator in the Fock space defined by

$$\mathcal{T}^0(Q) = \Pi Q : \mathcal{F}_n \to \mathcal{F}_n.$$

Bargmann transform

The Bargmann transform is an isometry $B : L^2(\mathbb{R}^n) \to \mathcal{F}_n$ defined by

$$Bf(z) = \pi^{-n/4} \int_{\mathbb{R}^n} \exp\left[-\left(\frac{1}{2}z\cdot z + \frac{1}{2}x\cdot x - \sqrt{2}z\cdot x\right)\right] f(x)dx, \quad z \in \mathbb{C}^n.$$

For the position and momentum operators in the Schrödinger presentation $\hat{q}_k = x_k, \quad \hat{p}_k = \frac{1}{i}\frac{\partial}{\partial x_k}$, the corresponding operators in the Bargmann-Fock presentation

$$B\hat{q}_k B^{-1} = \frac{1}{\sqrt{2}}\left(z_k + \frac{\partial}{\partial z_k}\right), \quad B\hat{p}_k B^{-1} = \frac{1}{\sqrt{2}} i \left(z_k - \frac{\partial}{\partial z_k}\right).$$

For the creation and annihilation operators $\hat{a}_k = \frac{1}{\sqrt{2}}(\hat{q}_k + i\hat{p}_k)$, $\hat{a}^*_k = \frac{1}{\sqrt{2}}(\hat{q}_k - i\hat{p}_k)$, the corresponding operators in the Bargmann-Fock presentation

$$B\hat{a}_k B^{-1} = \frac{\partial}{\partial z_k}, \quad B\hat{a}^*_k B^{-1} = z_k.$$

- For any $F \in L^2(\mathbb{C}^n; e^{-\frac{1}{2}|z|^2}dz)$, $\frac{\partial}{\partial z_k}\Pi F = \Pi \bar{z}_k F$;
- For any $F \in \mathcal{F}_n$, $z_k \Pi F = \Pi(z_k F)$.

Let P be a polynomial in $\mathbb{C}^n$. If we write P as

$$P(\bar{z}, z) = \sum_{k,l} A_{k;l} \bar{z}_1^{k_1} \dots \bar{z}_n^{k_n} z_1^{l_1} \dots z_n^{l_n},$$

then, for any $F \in \mathcal{F}_n$,

$$\mathcal{T}^0(P)F = \Pi(P(\bar{z}, z)F) = P(\partial_z, z)F,$$

where $P(\partial_z, z)$ is the operator in $\mathcal{F}_n$ given by

$$P(\partial_z, z) = \sum_{k,l} A_{k;l} \frac{\partial^{k_1}}{\partial z_1^{k_1}} \dots \frac{\partial^{k_n}}{\partial z_n^{k_n}} z_1^{l_1} \dots z_n^{l_n}.$$

Anti-Wick symbols (Berezin, 1971)

- Under the Bargmann transform B, the operator $\mathcal{T}^0(P)$ in $\mathcal{F}_n$ corresponds to the operator $B^{-1}\mathcal{T}^0(P)B$ in $L^2(\mathbb{R}^n)$ given by

$$B^{-1}\mathcal{T}^0(P)B = \sum A_{k;l}\hat{a}^{k_1}\dots\hat{a}^{k_n}(\hat{a}_1^*)^{l_1}\dots(\hat{a}_n^*)^{l_n}.$$

- The operator $B^{-1}\mathcal{T}^0(P)B$ is the differential operator with polynomial coefficients in $\mathbb{R}^n$;
- $P(\bar{z}, z)$ is the anti-Wick symbol of $B^{-1}\mathcal{T}^0(P)B$ (the Schrödinger presentation) and $\mathcal{T}^0(P)$ (the Bargmann-Fock presentation);
- Some sufficient conditions of self-adjointness of the operator $B^{-1}P(\partial_z, z)B$ (Berezin, 1971).

One can compute the Weyl symbol of this operator by the well-known formula (Berezin, 1971). If P is a positive definite quadratic form, then

$$B^{-1}\mathcal{T}^0(P)B = Op_w(\tilde{P}) + \frac{\mathrm{tr}(\tilde{P})}{2},$$

where $\tilde{P}$ is a quadratic form on $\mathbb{R}^{2n}$, corresponding to P under the linear isomorphism $(x,\xi)\in\mathbb{R}^{2n}\cong T^*\mathbb{R}^n \mapsto z\in\mathbb{C}^n$:

$$z_k = \frac{1}{\sqrt{2}}(x_k - i\xi_k), \quad k = 1,\ldots,n.$$

$Op_w(\tilde{P})$ is the pseudodifferential operator in $\mathbb{R}^n$ with Weyl symbol $\tilde{P}$.

The Bochner-Laplacian Δ^{L^p}:

$$\Delta^{L^p} = (\nabla^{L^p})^* \nabla^{L^p} : C^\infty(X, L^p) \to C^\infty(X, L^p).$$

τ is a smooth function on X:

$$\tau(x) = \pi \operatorname{Tr}[(-J_0^2(x))^{1/2}], \quad x \in X.$$

For the magnetic Laplacian $\Delta^L = -\sum_{j=1}^d \left(\frac{\partial}{\partial z_j} - iA_j(z)\right)^2$

$$\tau(x) = \frac{1}{2} \operatorname{Tr}(B^* B)^{1/2} = \operatorname{Tr}^+(B).$$

Assumptions

- $\min_{x \in X} \tau(x) = \tau_0$;
- There exists a unique $x_0 \in X$ such that $\tau(x) = \tau_0$, which is non-degenerate:

$$\operatorname{Hess} \tau(x_0) > 0.$$

Schrödinger operator type representation

$$\Delta^{L^p} = (\nabla^{L^p})^* \nabla^{L^p} = \Delta_p + p\tau,$$

the renormalized Bochner-Laplacian Δ_p satisfies the gap property:

$$\sigma(\Delta_p) \subset [-C_L, C_L] \cup [2p\mu_0 - C_L, +\infty),$$

$\mathcal{H}_p$ corresponds to the lowest Landau levels.

Upper estimates for $\lambda_j(\Delta^{L^p})$ (the Rayleigh-Ritz technique):

$$\lambda_j(\Delta^{L^p}) \leqslant \lambda_j(P_{\mathcal{H}_p} \Delta^{L^p} P_{\mathcal{H}_p}), \quad j \in \mathbb{N}.$$

The operator

$$p^{-1}P_{\mathcal{H}_p}\Delta^{L^p}P_{\mathcal{H}_p}=p^{-1}P_{\mathcal{H}_p}\Delta_pP_{\mathcal{H}_p}+P_{\mathcal{H}_p}\tau P_{\mathcal{H}_p}$$

is a Toeplitz operator:

$$p^{-1}P_{\mathcal{H}_p}\Delta^{L^p}P_{\mathcal{H}_p}=P_{\mathcal{H}_p}\left(\sum_{l=0}^{\infty}p^{-l}g_l\right)P_{\mathcal{H}_p}+\mathcal{O}(p^{-\infty}).$$

The leading term

$$g_0(x)=\tau(x).$$

The next term is the principal symbol of $\Delta_pP_{\mathcal{H}_p}$:

$$g_1(x)=J_{1,2}(x).$$

Computation of $J_{1,2}$

Put

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial Z_{2j-1}} - i\frac{\partial}{\partial Z_{2j}}\right), \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial Z_{2j-1}} + i\frac{\partial}{\partial Z_{2j}}\right).$$

Let $\mathcal{R}(Z) = \sum_{j=1}^{2n} Z_j e_j = Z$ denote the radial vector field on $T_{x_0}X$. Define first order differential operators $b_j, b_j^+, j = 1, \ldots, n$, on $T_{x_0}X$ by

$$b_j = -2\nabla_{\frac{\partial}{\partial z_j}} - R^L_{x_0}(\mathcal{R}, \tfrac{\partial}{\partial z_j}) \quad b_j^+ = 2\nabla_{\frac{\partial}{\partial \overline{z}_j}} + R^L_{x_0}(\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_j}).$$

So we can write

$$\mathcal{L}_{x_0} = -\sum_{j=1}^{2n}\left(\frac{\partial}{\partial e_j} + \frac{1}{2}R^L_{x_0}(Z, e_j)\right)^2 - \tau(x_0) = \sum_{j=1}^{n} b_j b_j^+.$$

$$J_{1,2}(x_0) = \frac{F_{1,2,x_0}(0,0)}{\mathcal{P}_{x_0}(0,0)}, \quad F_{1,2,x_0}(Z,Z') = [\mathcal{P}_{x_0}\mathcal{F}_{1,2,x_0}\mathcal{P}_{x_0}](Z,Z'),$$

where $\mathcal{F}_{1,2,x_0}$ is an unbounded linear operator in $L^2(T_{x_0}X)$ given by

$$\begin{aligned}\mathcal{F}_{1,2,x_0} = 4 &\left\langle R^{TX}_{x_0}\left(\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial z_j}\right)\tfrac{\partial}{\partial \overline{z}_i}, \tfrac{\partial}{\partial \overline{z}_j}\right\rangle \\ &+ \left\langle (\nabla^X\nabla^X \mathcal{J})_{(\mathcal{R},\mathcal{R})}\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial \overline{z}_i}\right\rangle + \frac{\sqrt{-1}}{4}\operatorname{tr}|_{TX}\left(\nabla^X\nabla^X(J\mathcal{J})\right)_{(\mathcal{R},\mathcal{R})} \\ &+ \frac{1}{9}|(\nabla^X_{\mathcal{R}}\mathcal{J})\mathcal{R}|^2 + \frac{4}{9}\left\langle (\nabla^X_{\mathcal{R}}\mathcal{J})\mathcal{R}, \tfrac{\partial}{\partial z_i}\right\rangle b_i^+ \mathcal{L}_0^{-1} b_i \left\langle (\nabla^X_{\mathcal{R}}\mathcal{J})\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i}\right\rangle.\end{aligned}$$

If $J_0 = J$ (almost-Kähler), then

$$J_{1,2}(x_0) = \frac{1}{24}|\nabla^X J|^2_{x_0}.$$

Here if $\{e_j\}$ is a local orthonormal frame of (TX, g^{TX}), then

$$|\nabla^X J|^2 = \sum_{ij} |(\nabla^X_{e_i} J)e_j|^2.$$

Upper bounds for eigenvalues

Consider the Toeplitz operator $\mathcal{T}_{x_0}$ in $L^2(T_{x_0}X)$ defined by

$$\mathcal{T}_{x_0} = \mathcal{P}_{x_0}(q_{x_0}(Z) + J_{1,2}(x_0))\mathcal{P}_{x_0}.$$

where

$$q_{x_0}(Z) = \left(\frac{1}{2}\mathrm{Hess}\,\tau(x_0)Z, Z\right), \quad Z \in T_{x_0}X.$$

Let $\{\lambda_j(\Delta^{L^p})\}$ be the increasing sequence of the eigenvalues of the operator Δ^{L^p} (counted with multiplicities) and $\{\mu_j\}$ be the increasing sequence of the eigenvalues of $\mathcal{T}_{x_0}$ (counted with multiplicities).

Theorem (Yu. K. (2018))

For any $j \in \mathbb{N}$, there exists $\phi_j \in \mathbb{R}$ such that

$$\lambda_j(\Delta^{L^p}) \leqslant p\tau_0 + \mu_j + p^{-1/2}\phi_j + \mathcal{O}(p^{-1}), \quad p \to \infty.$$

2D-magnetic Laplacian

In particular, for the 2D-magnetic Laplacian on a Riemann surface X:

$$\mathbf{B} = b(x)dv_g, \quad dv_g = \sqrt{g}dx_1 \wedge dx_2,$$

assume $b(x) > 0$ for any $x \in X$ and there exists a unique x_0 such that

$$b(x_0) = b_0 := \min_{x\in X} b(x),$$

if we denote

$$a = \mathrm{Tr}\left(\frac{1}{2}\mathrm{Hess}\, b(x_0)\right)^{1/2}, \quad d = \det\left(\frac{1}{2}\mathrm{Hess}\, b(x_0)\right).$$

we have (Helffer-Y.K., 2010, Helffer-Morame, 2001):

$$\lambda_j(\Delta^{L^p}) \leqslant pb_0 + \left[\frac{2d^{1/2}}{b_0}j + \frac{a^2}{2b_0}\right] + C_j p^{-1/2}, \quad j \in \mathbb{N}.$$