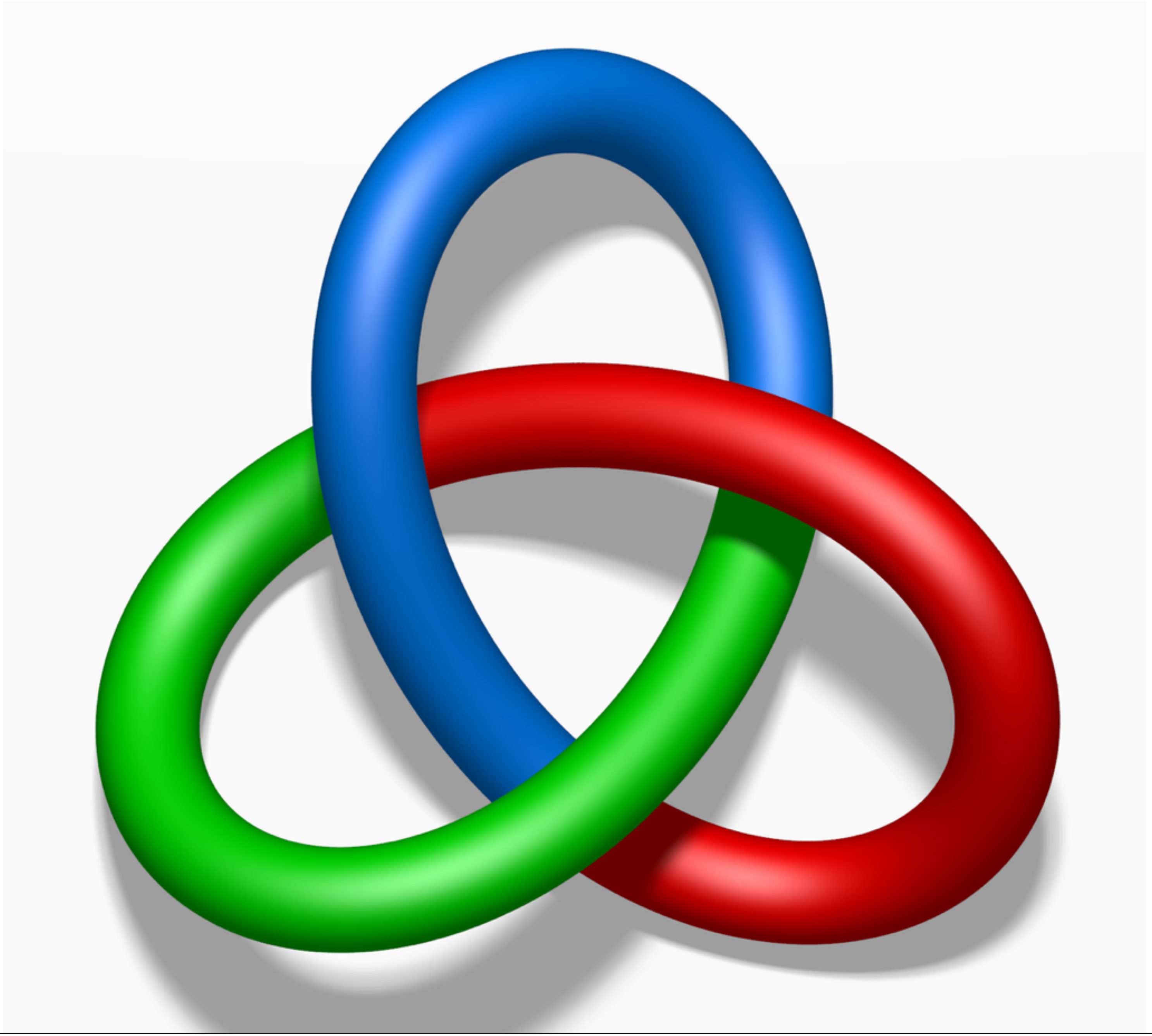
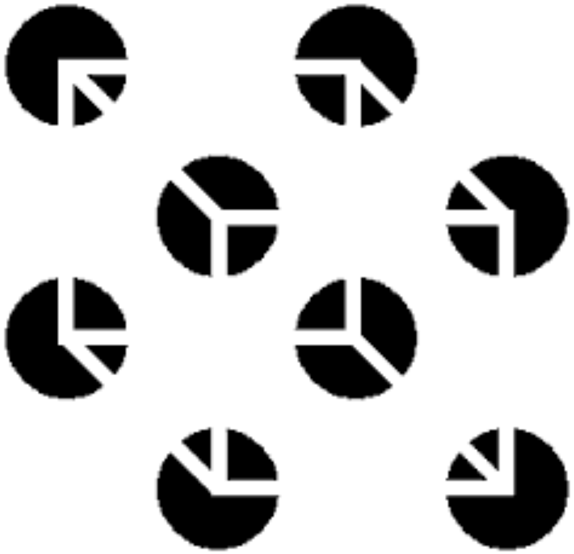
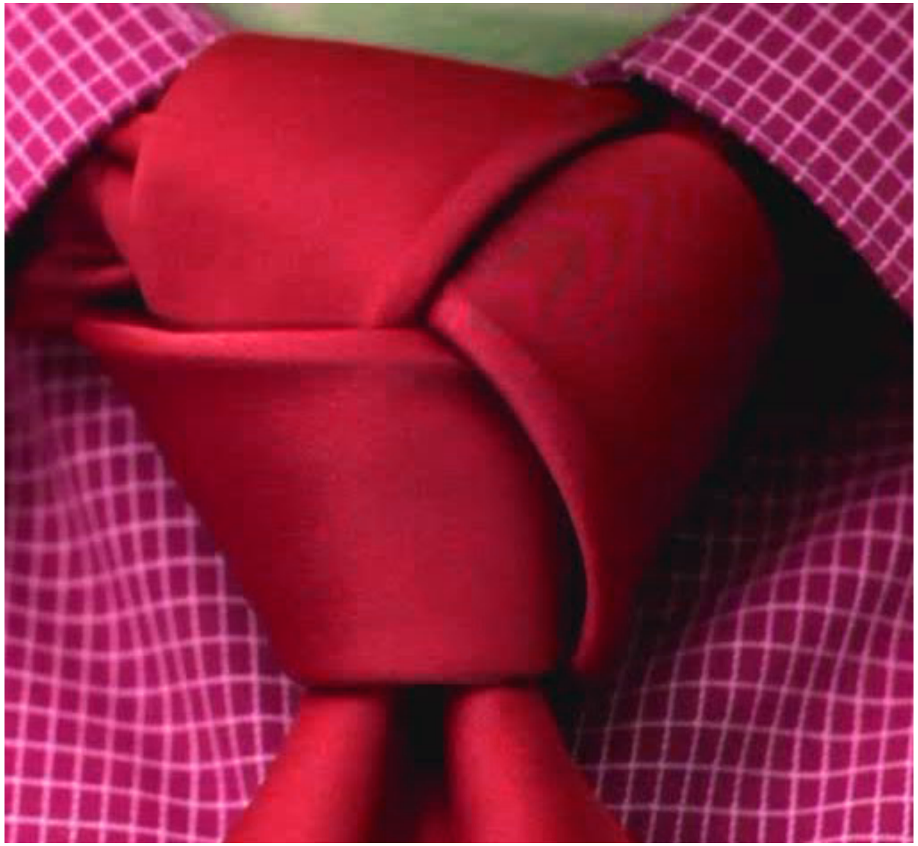


Virtual Knot Cobordism

Louis H Kauffman
UIC





Virtual Knot Theory
studies stabilized knots in thickened surfaces.

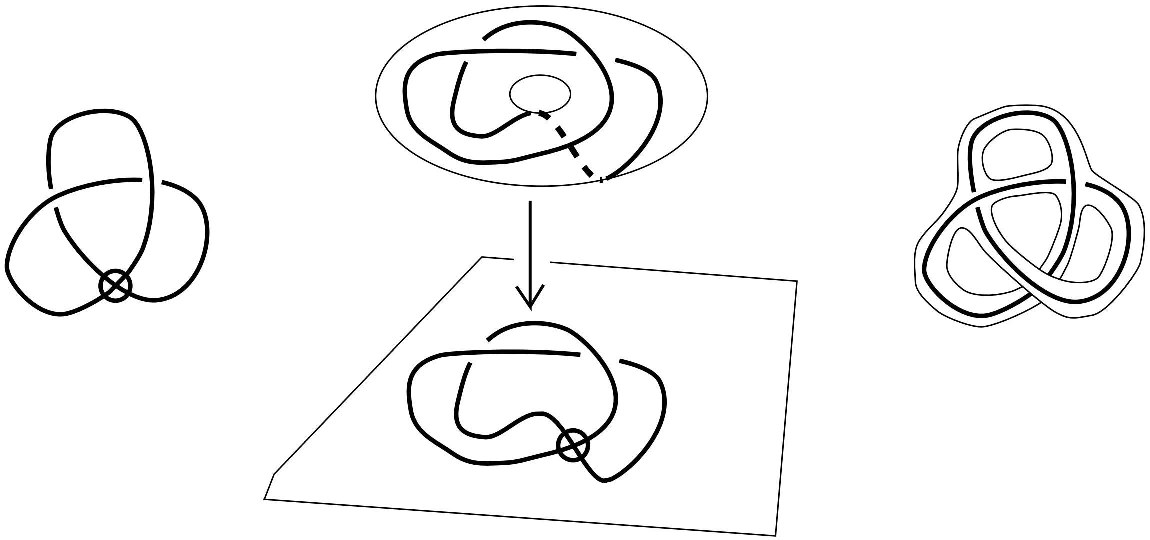
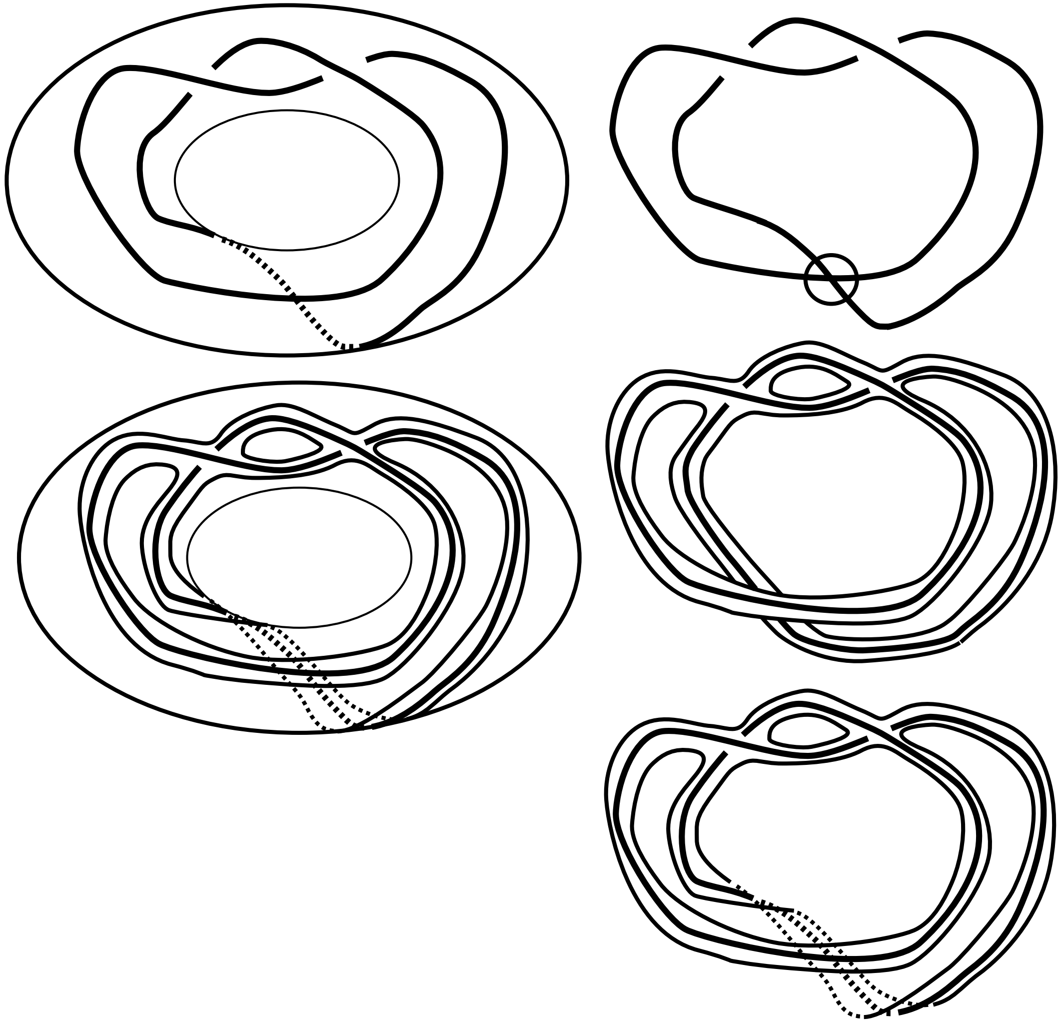
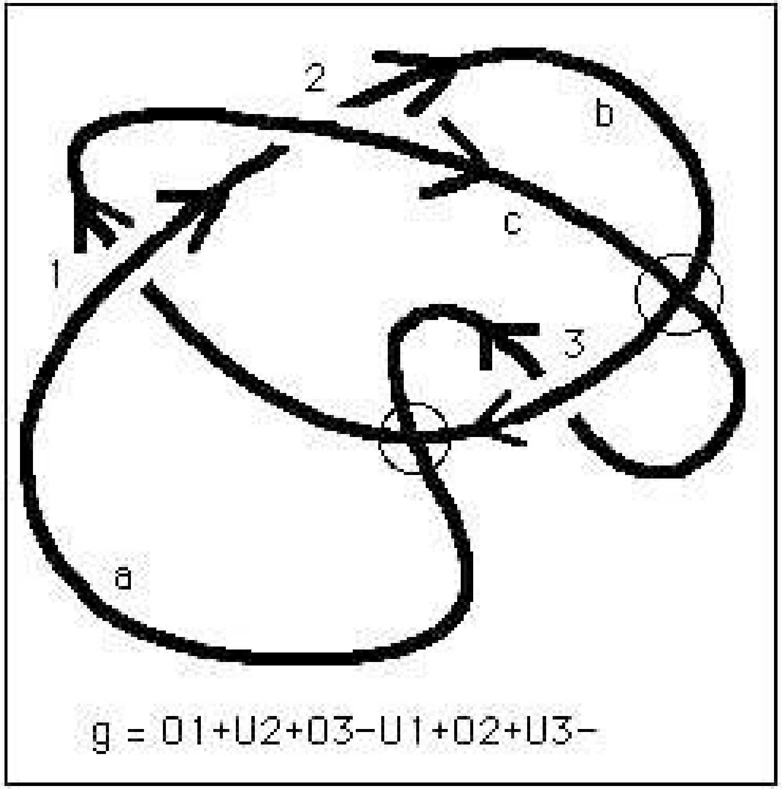


Figure 4: **Surfaces and Virtuals**



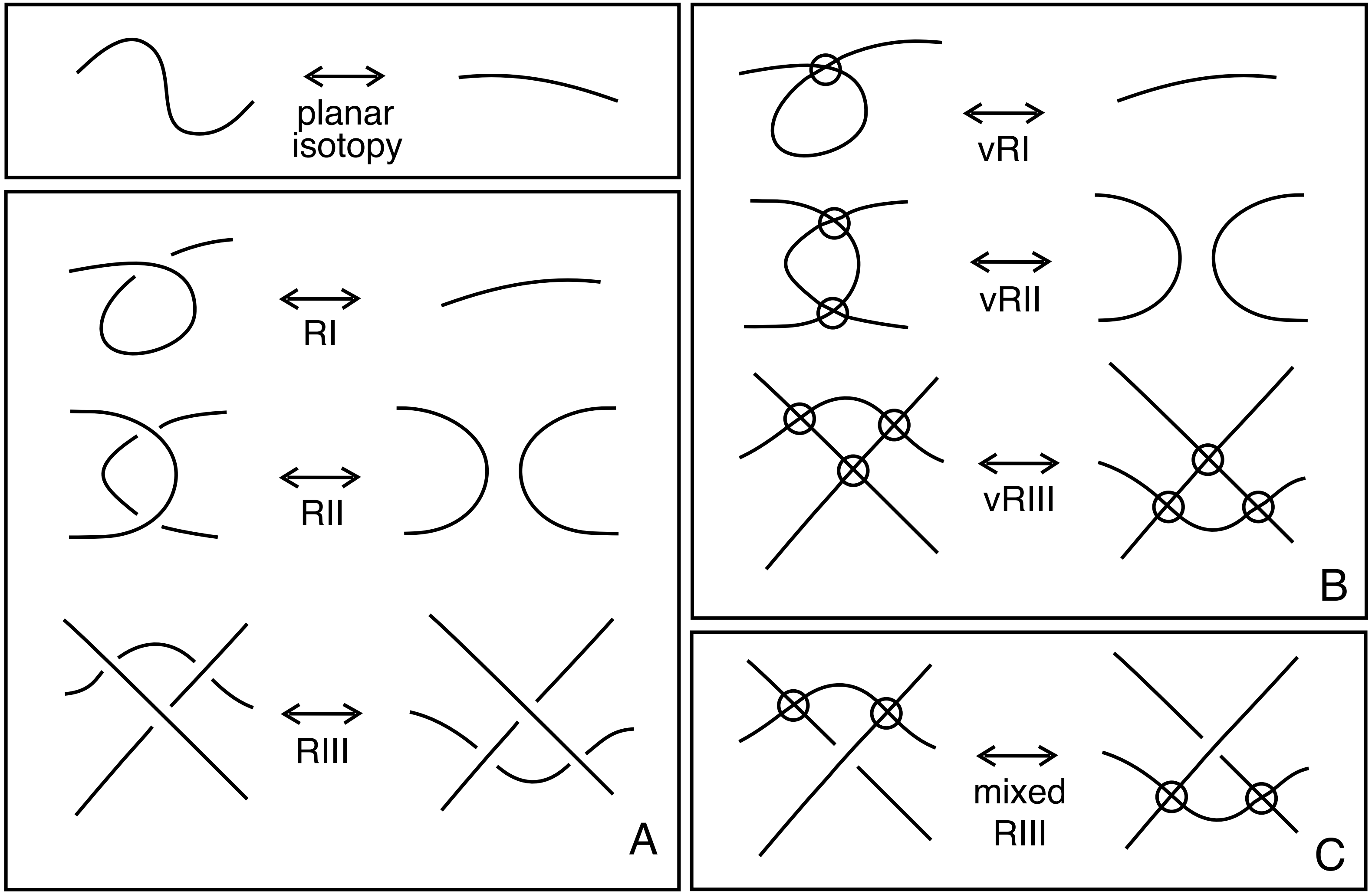


Virtual knots are all oriented (signed) Gauss codes taken up to Reidemeister moves on the codes.

Virtual crossings are artifacts of the planar diagram.

$$g = O1 + U2 + O3 - U1 + O2 + U3 - .$$

Generalized Reidemeister Moves for Virtual Knots and Links



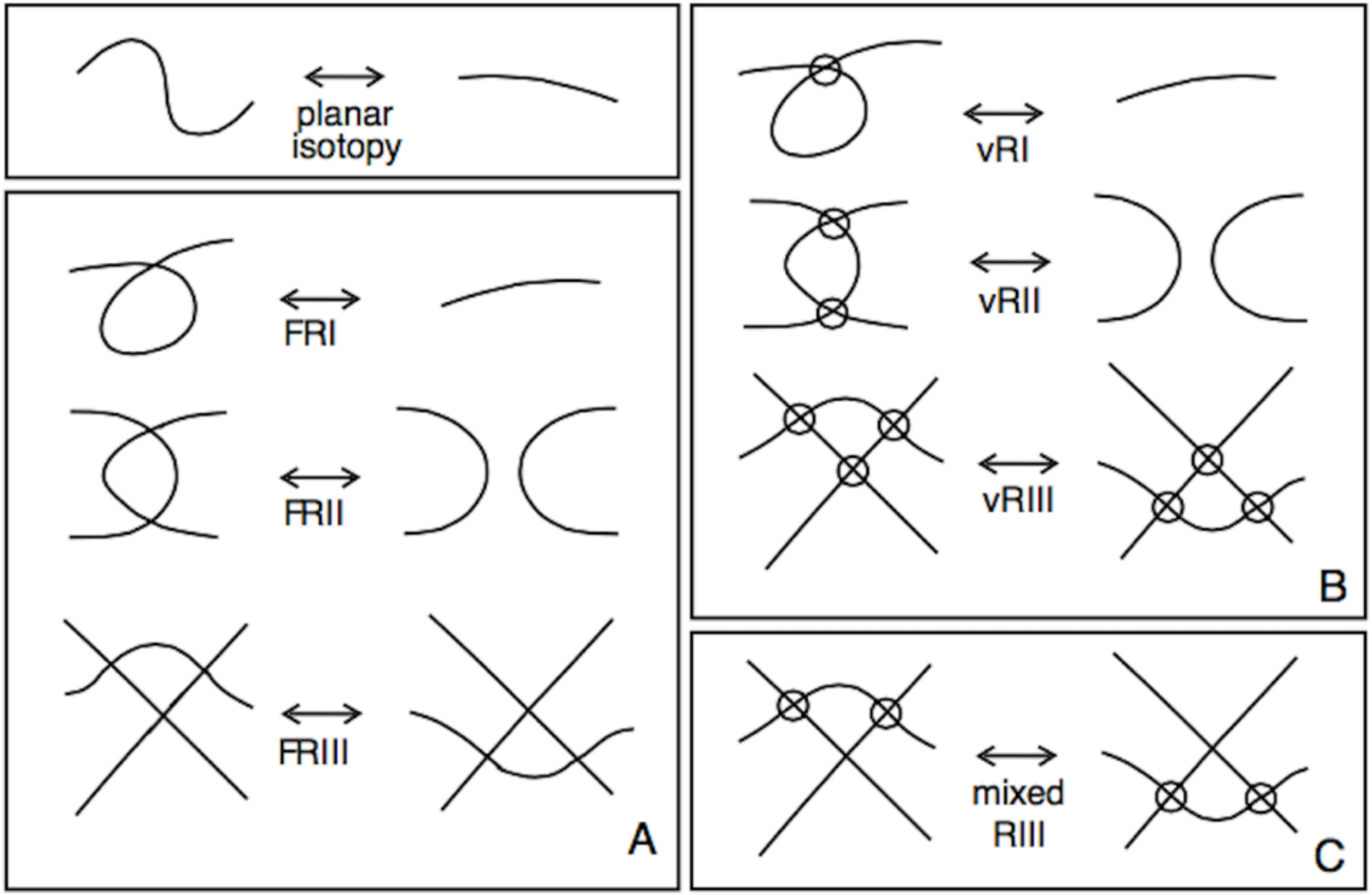


Figure 2: **Flat Virtual Moves**

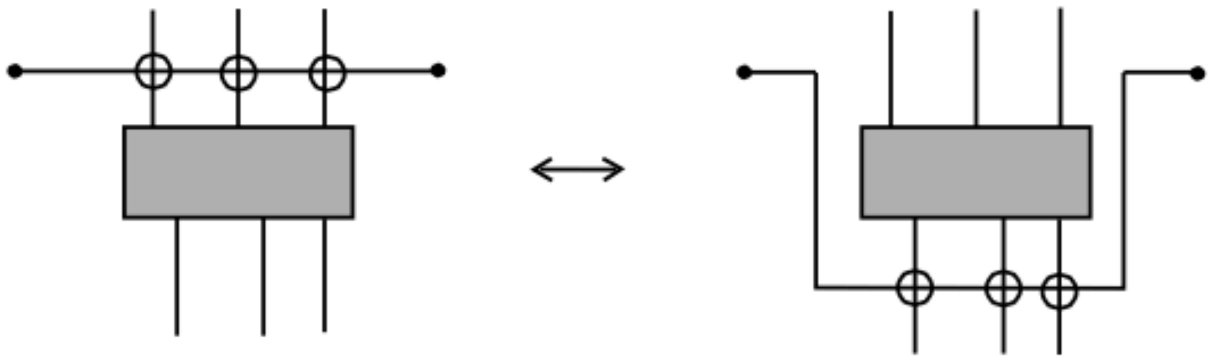


Figure 3: **Detour Move**

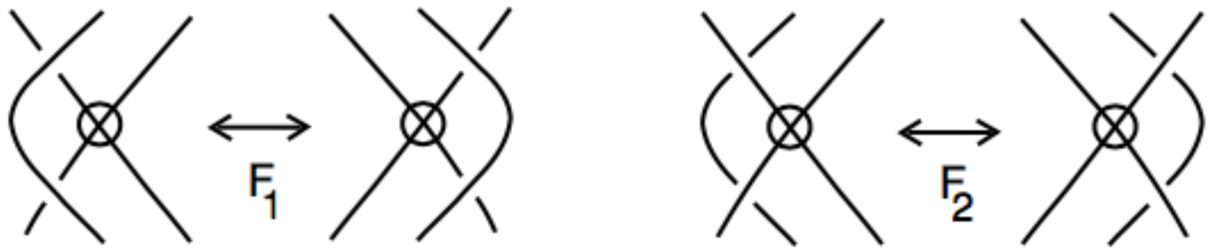


Figure 4: **Forbidden Moves**

The bracket polynomial [18] model for the Jones polynomial [14, 15, 16, 42] is usually described by the expansion

$$\langle \times \rangle = A \langle \asymp \rangle + A^{-1} \langle) (\rangle \tag{1}$$

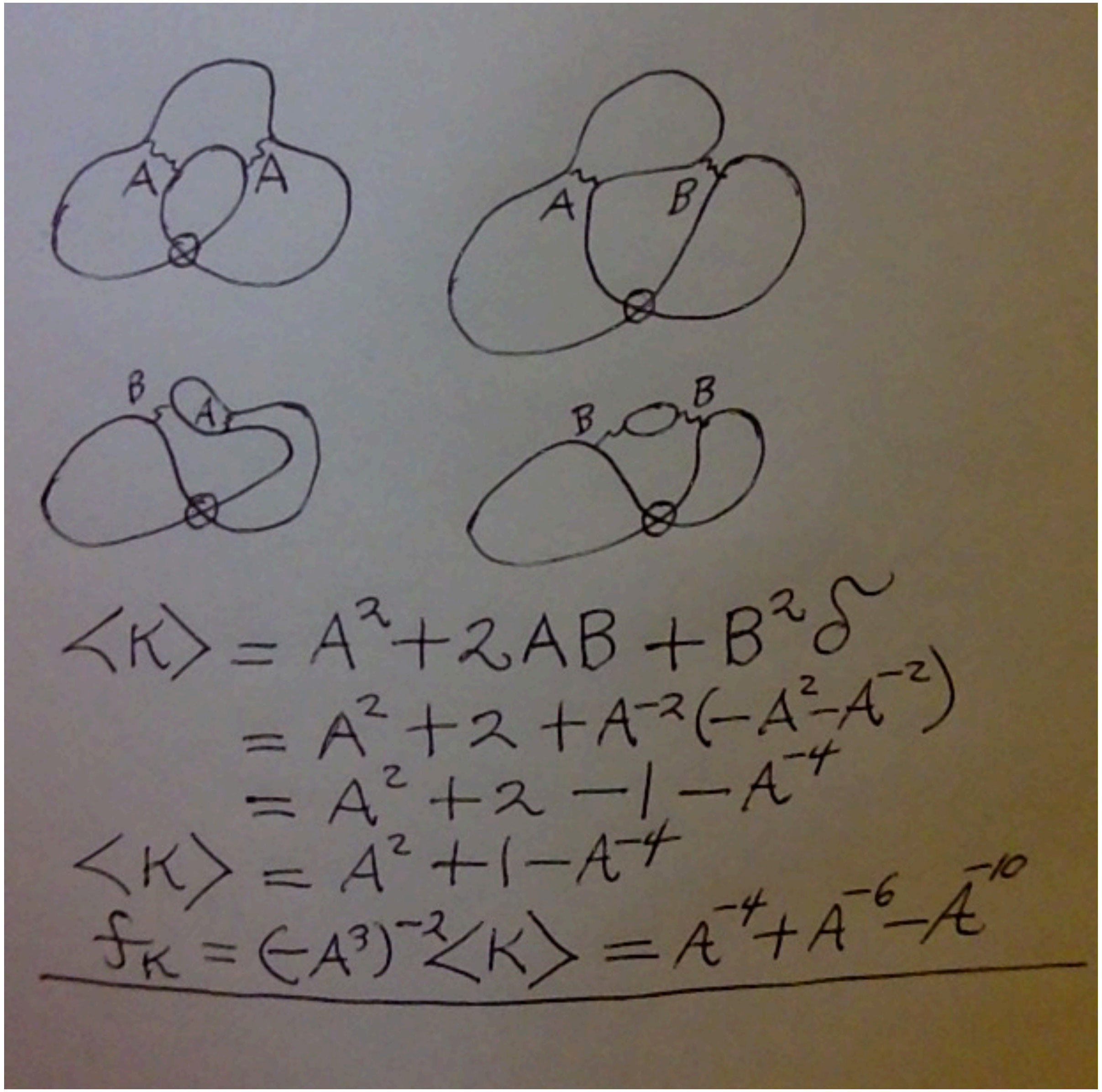
and we have

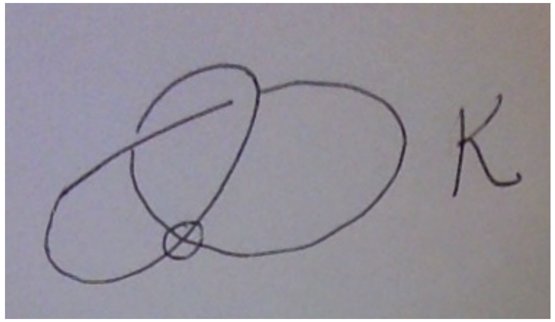
$$\langle K \bigcirc \rangle = (-A^{2} - A^{-2}) \langle K \rangle \tag{2}$$

$$\langle \partial \rangle = (-A^{3}) \langle \smile \rangle \tag{3}$$

$$\langle \partial \rangle = (-A^{-3}) \langle \smile \rangle \tag{4}$$

We call a diagram in the plane *purely virtual* if the only crossings in the diagram are virtual crossings. Each purely virtual diagram is equivalent by the virtual moves to a disjoint collection of circles in the plane.



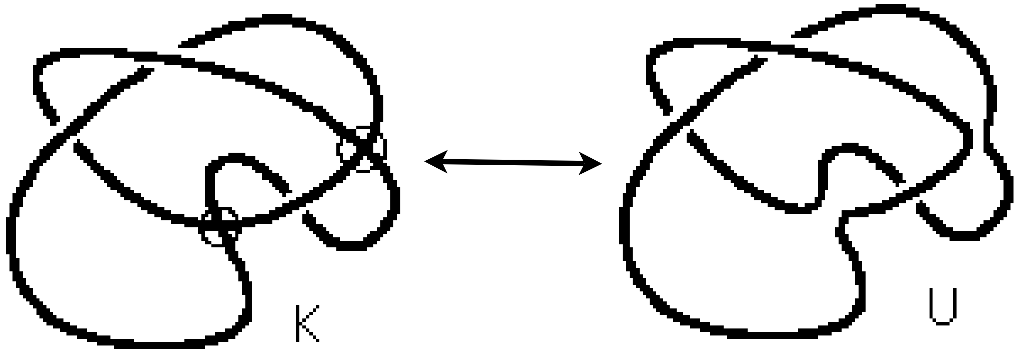


K is non-trivial, non-classical and chiral.

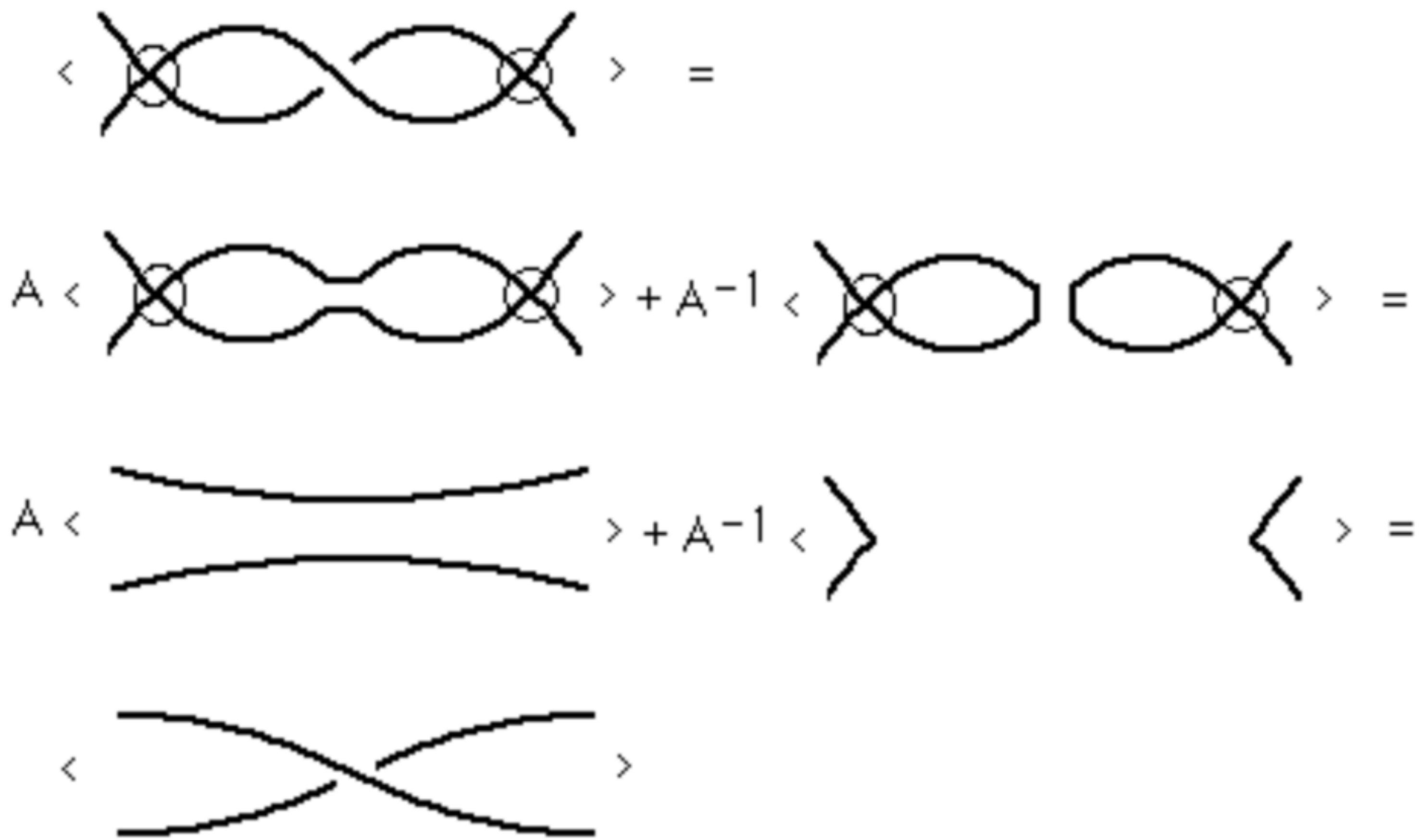
There exist infinitely many non-trivial K
with unit Jones polynomial.

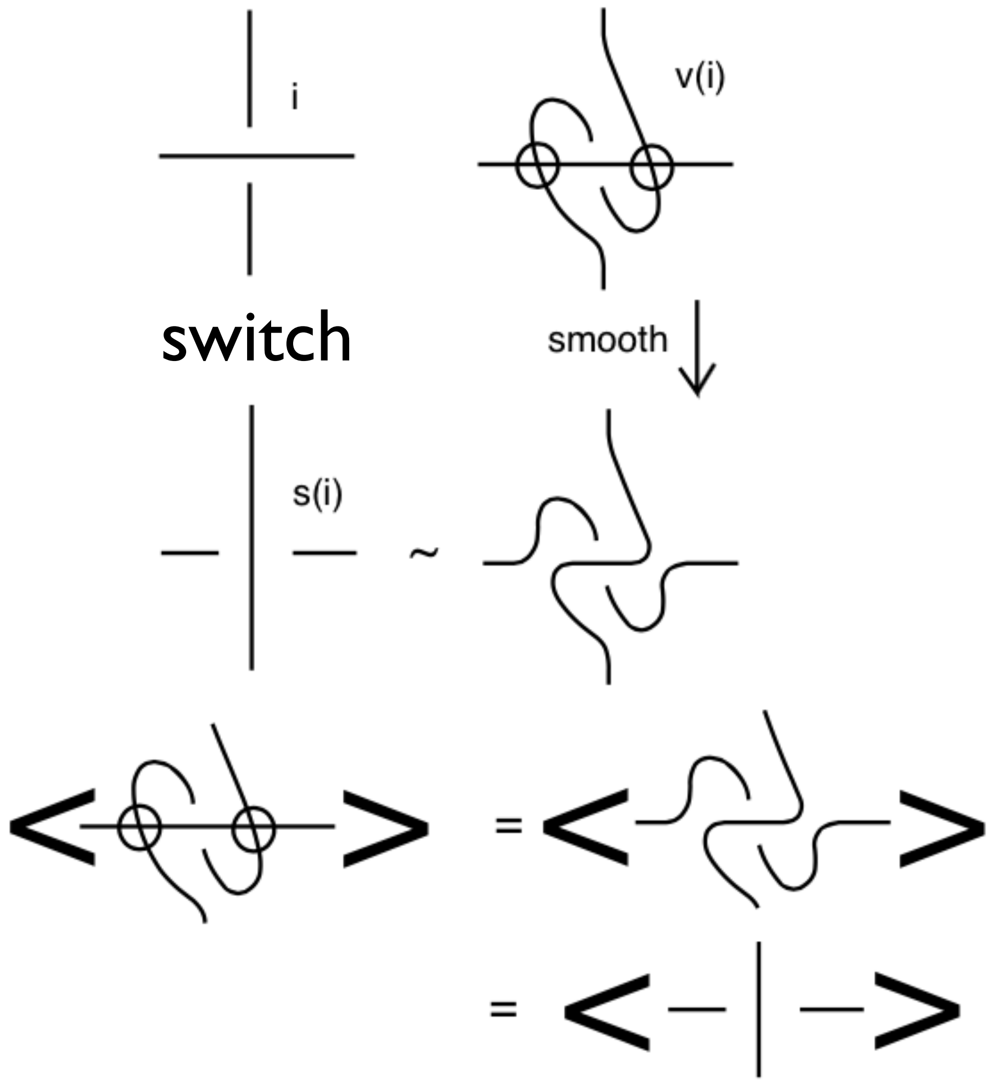
Bracket Polynomial is Unchanged
when smoothing flanking virtuals.

Z-Equivalence



Bracket Polynomial is Unchanged when smoothing flanking virtuals.




i
v(i)
switch
smooth
s(i)
~
=
=

Virtualization does not change the IQ(K).

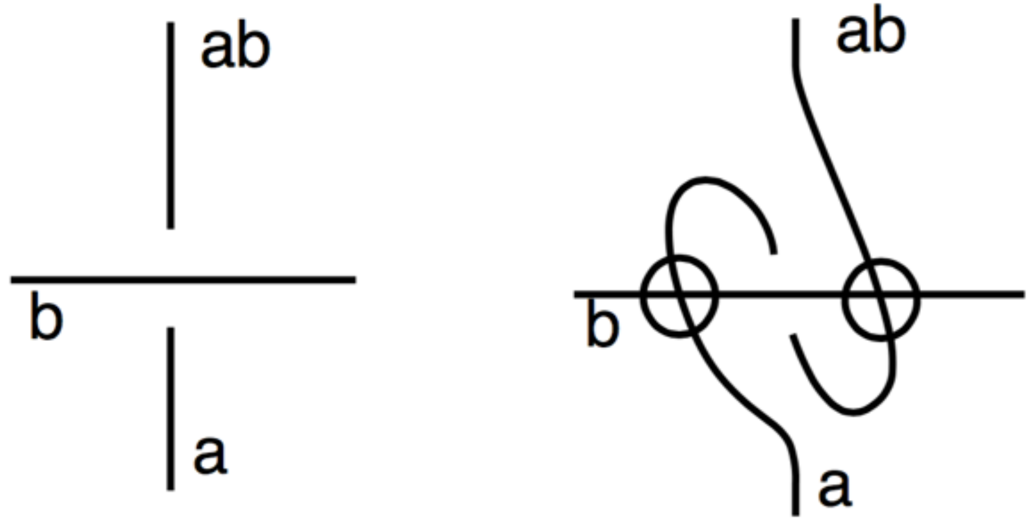
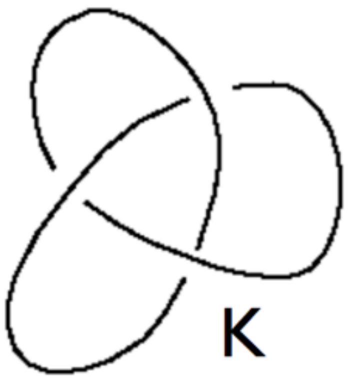


Figure 8. IQ(Virt)

The composition ab can denote a group theoretic operation
For example, let ab = b.a^(-1).b where a.b is group multiplication. The resulting group presentation is, for classical knots, the fundamental group of the two-fold branched covering along the knot.



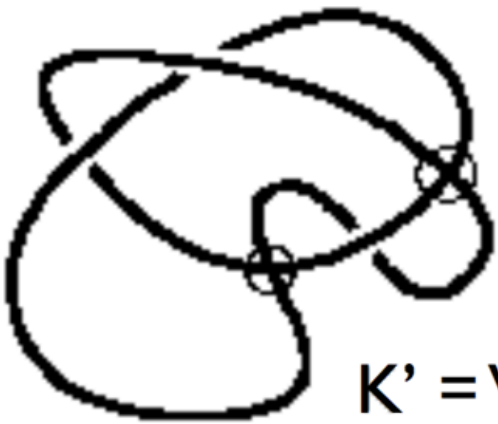
K

<Virt(K)> = <Switch(K)>

and

IQ(Virt(K)) = IQ(K).

Conclusion: There exist infinitely many non-trivial Virt(K) with unit Jones polynomial.



K’ = Virt(K)

U

Virtual Knot Cobordism

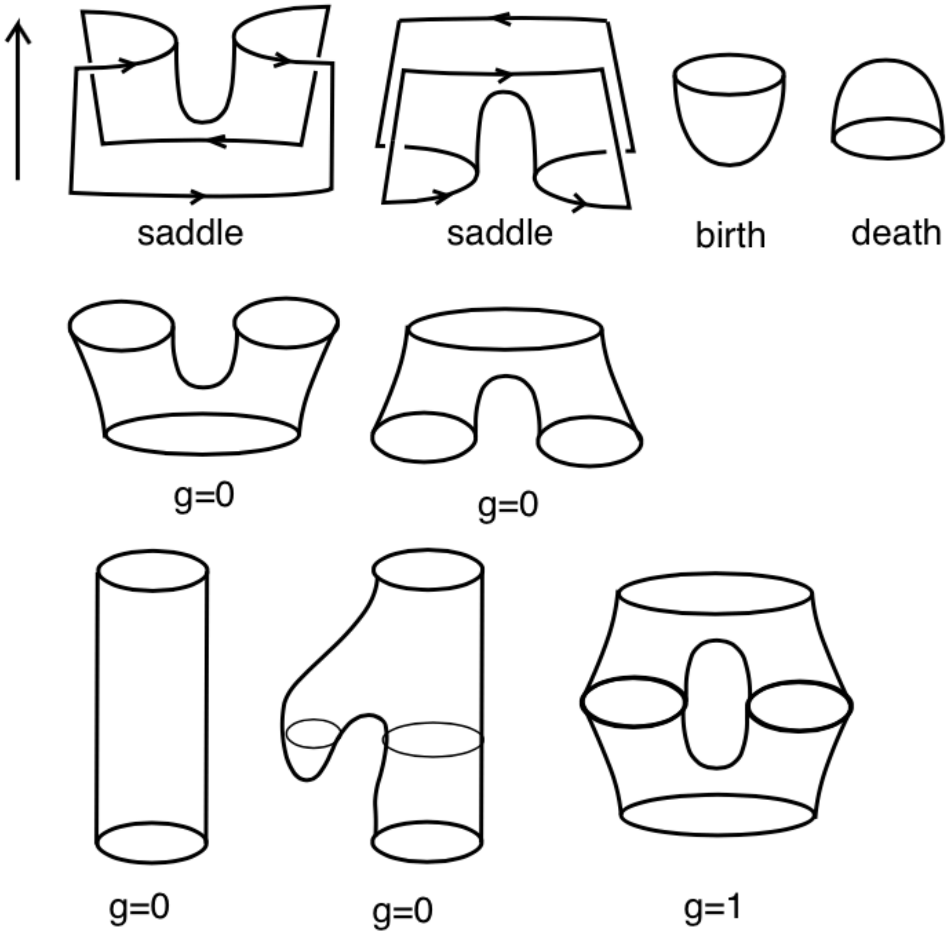


Figure 16: **Saddles, Births and Deaths**

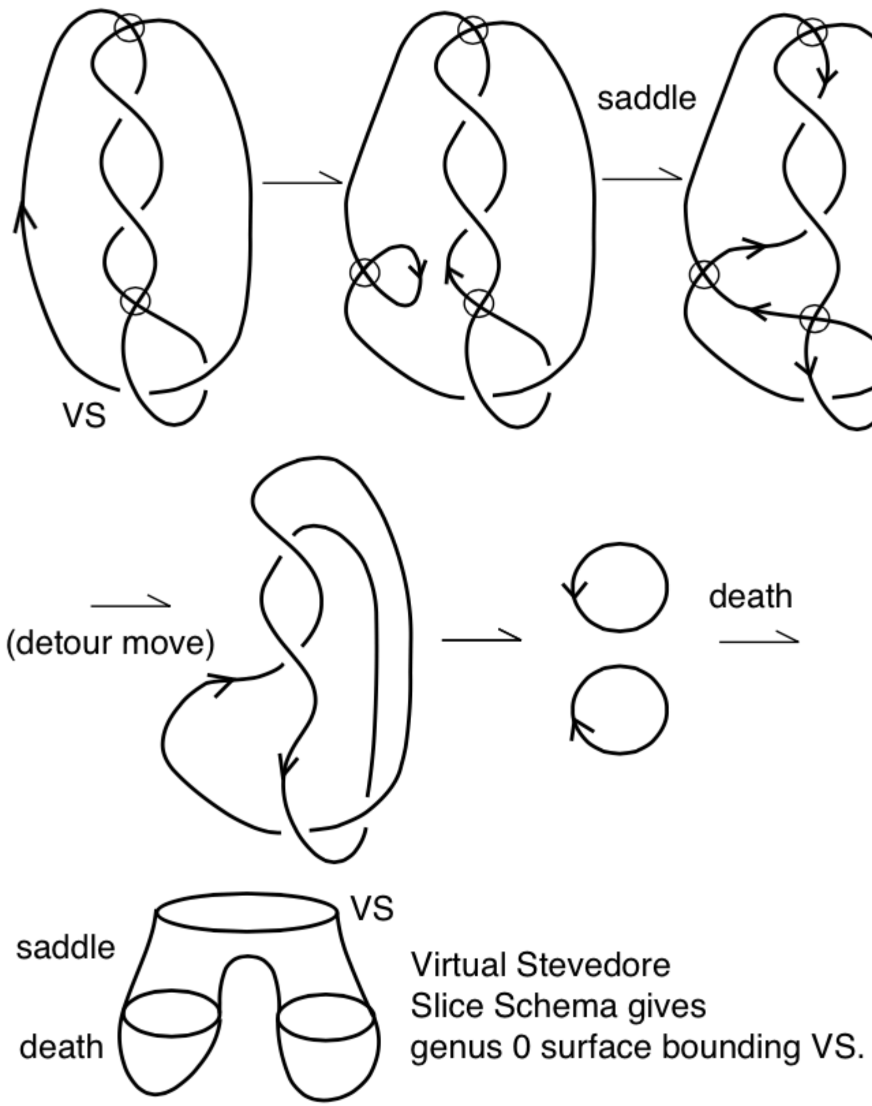
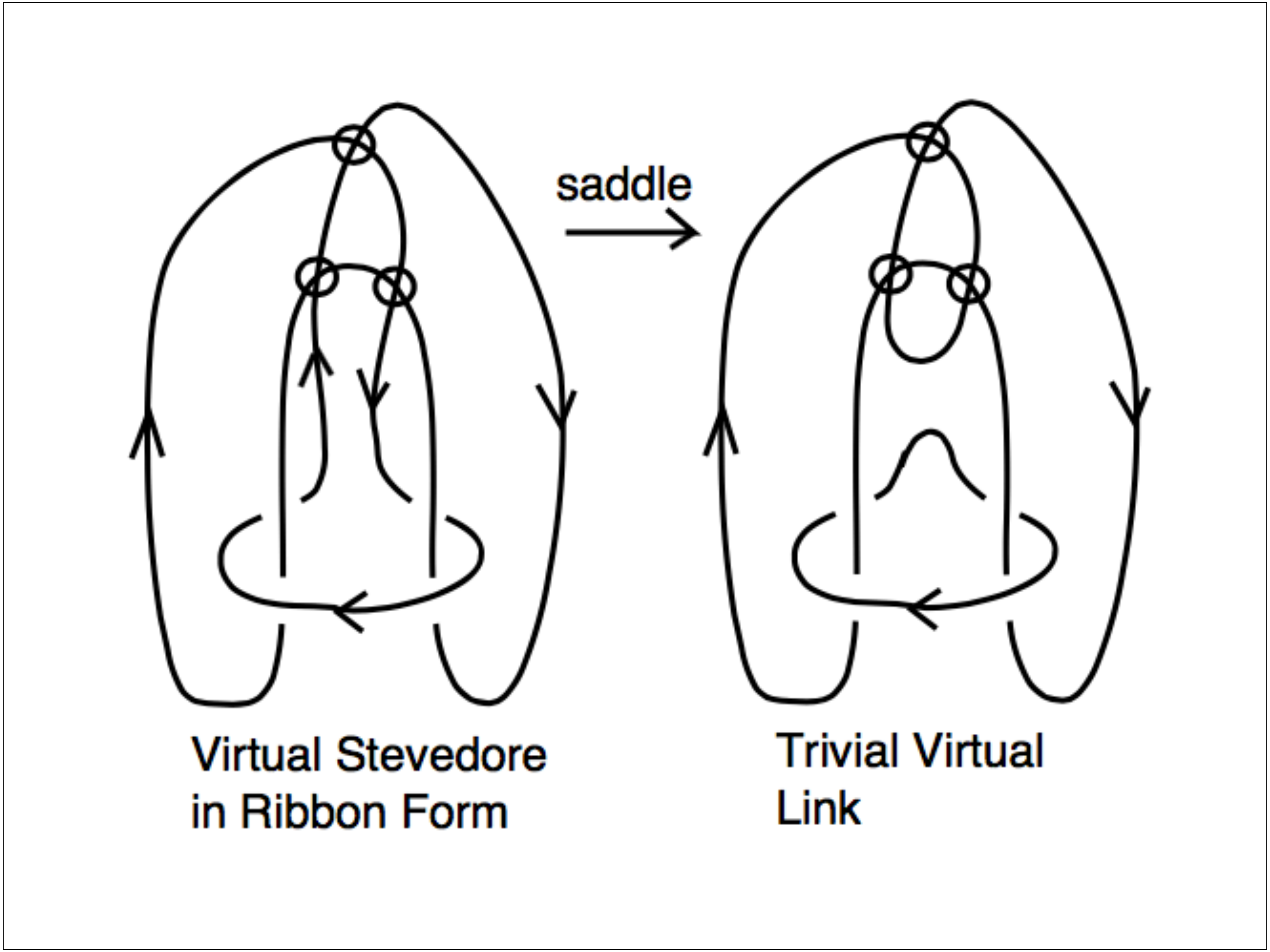
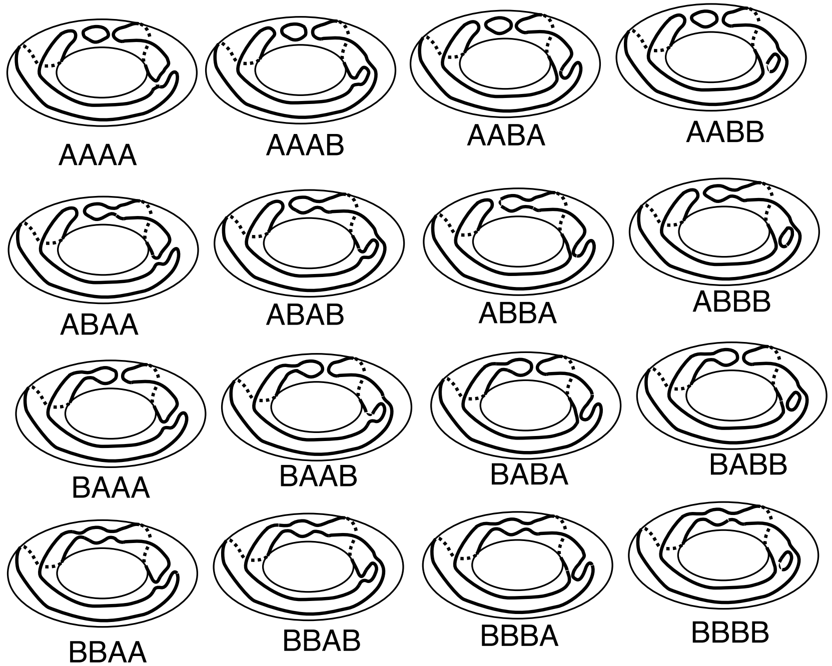
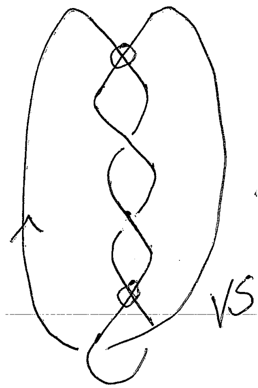


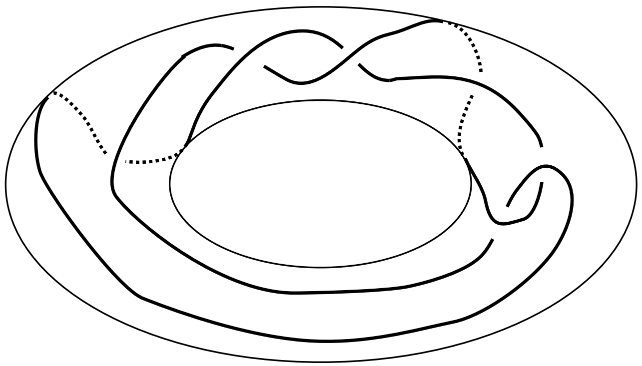
Figure 17: **Virtual Stevedore is Slice**


saddle
Virtual Stevedore
in Ribbon Form
Trivial Virtual
Link

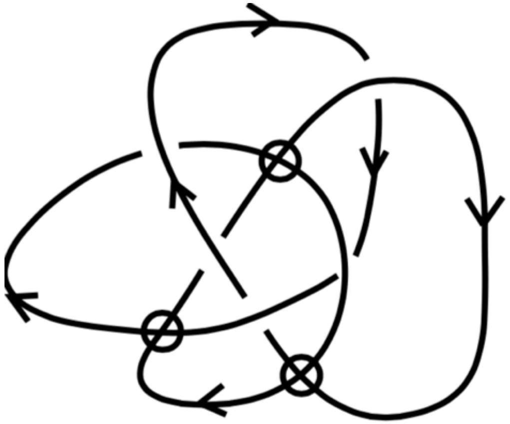
Virtual Stevedore is not classical.



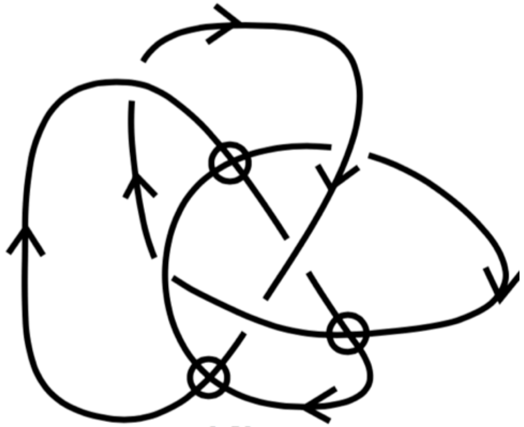




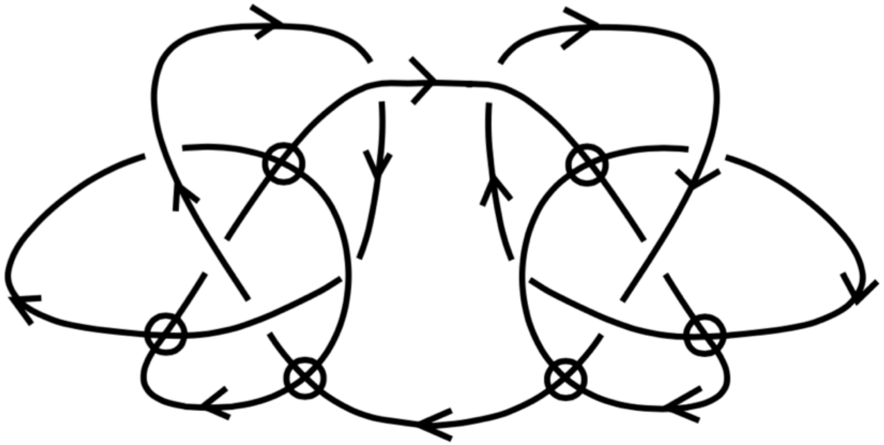
Vertical Mirror Image



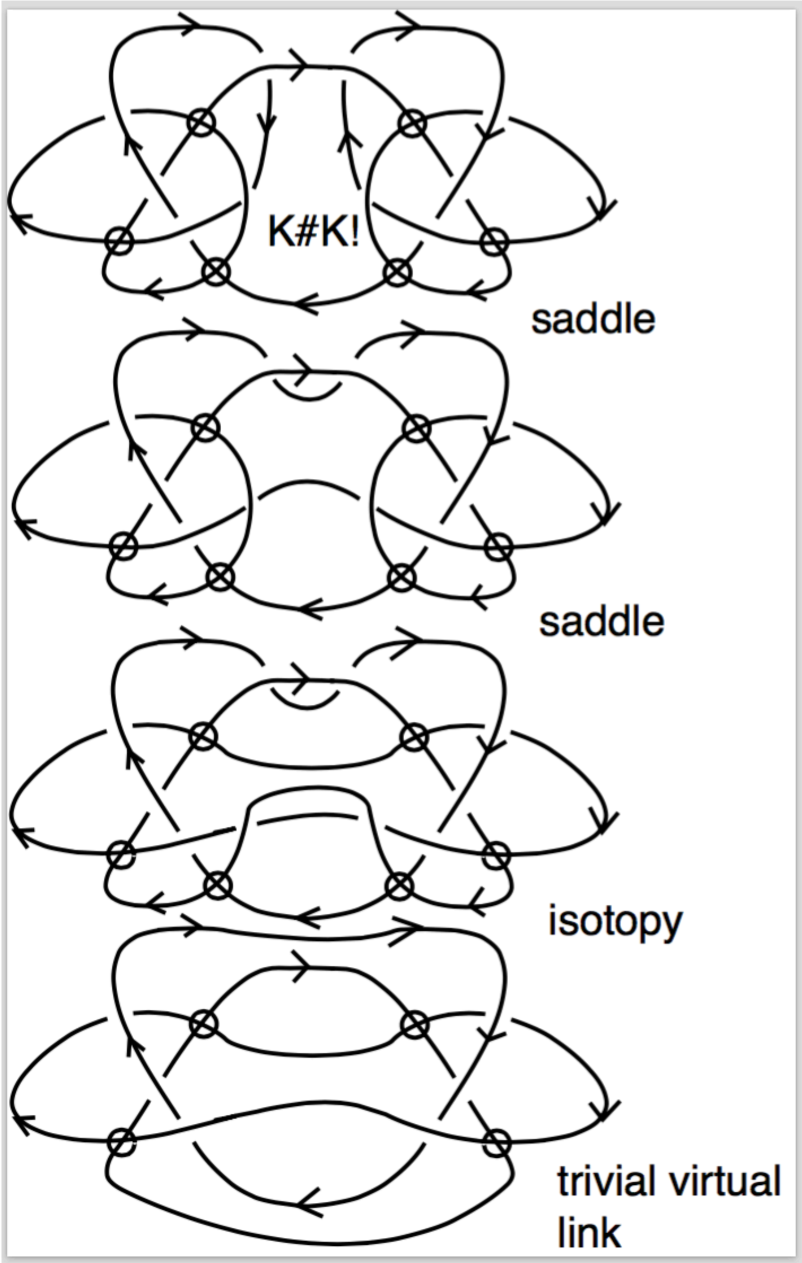
K



K!



K#K!



Connected Sum
with the
Vertical Mirror Image
is
Slice.

We say that K is concordant to K`

$K \sim_c K`$

if there exists a cobordism from K to K` of genus 0.

A virtual knot is said to be slice
if it is concordant to the unknot.

Spanning Surfaces for Knots and Virtual Knots.

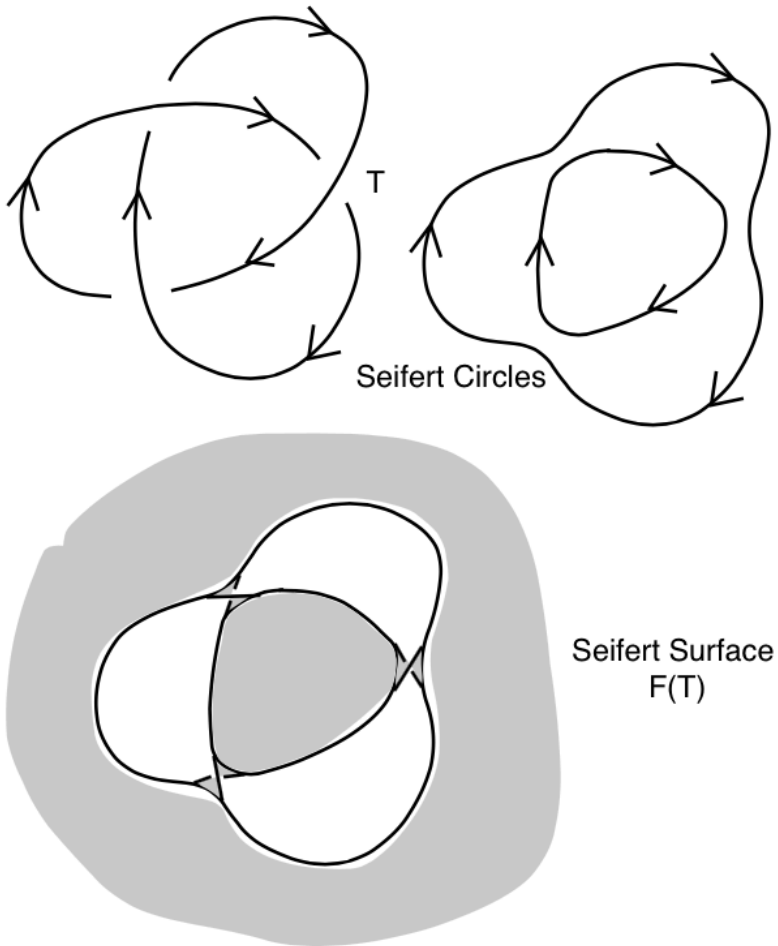
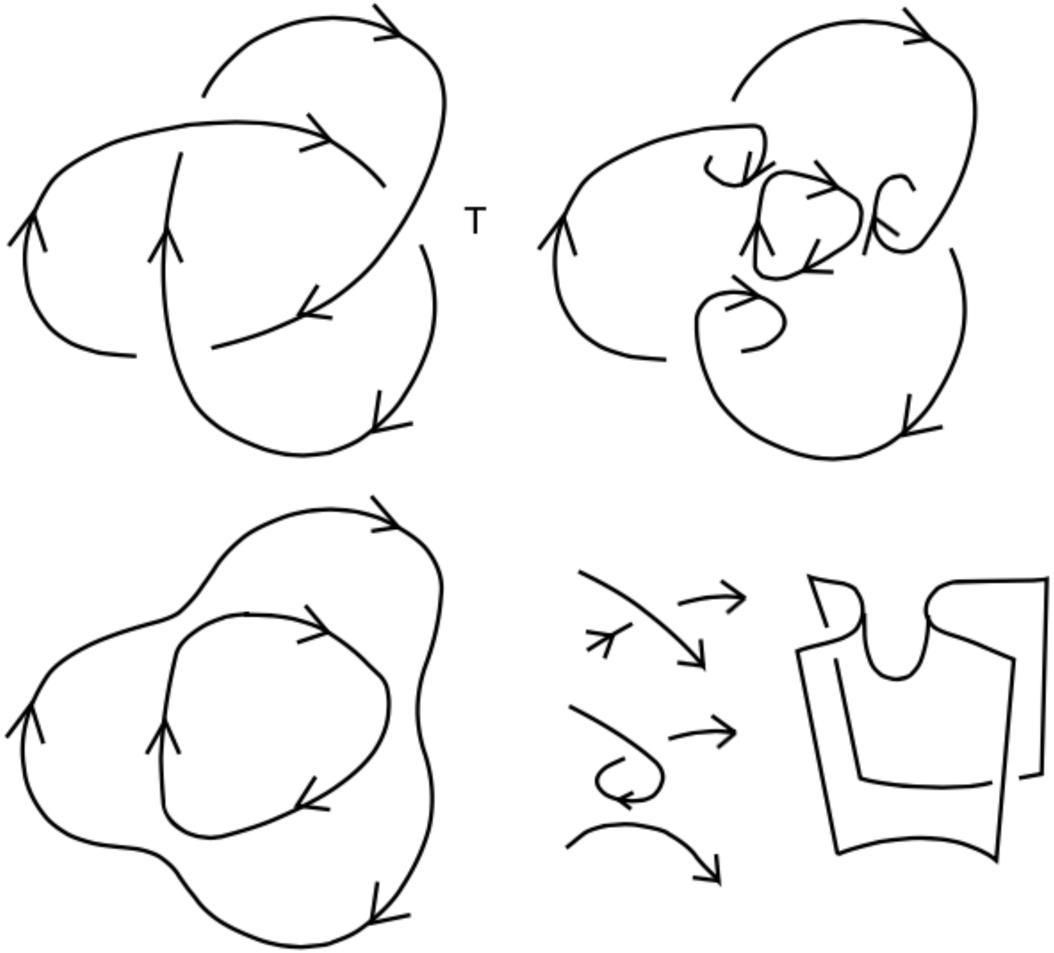
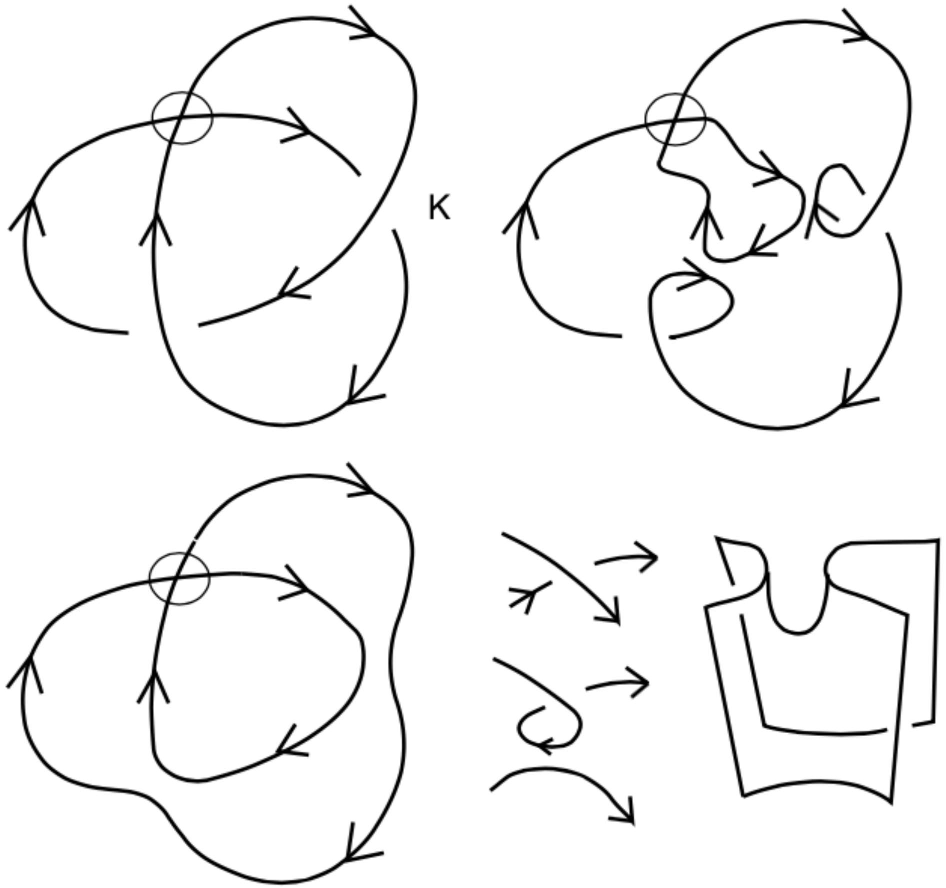


Figure 18: **Classical Seifert Surface**



Every classical knot diagram bounds a surface in the four-ball whose genus is equal to the genus of its Seifert Surface.

Figure 19: **Classical Cobordism Surface**

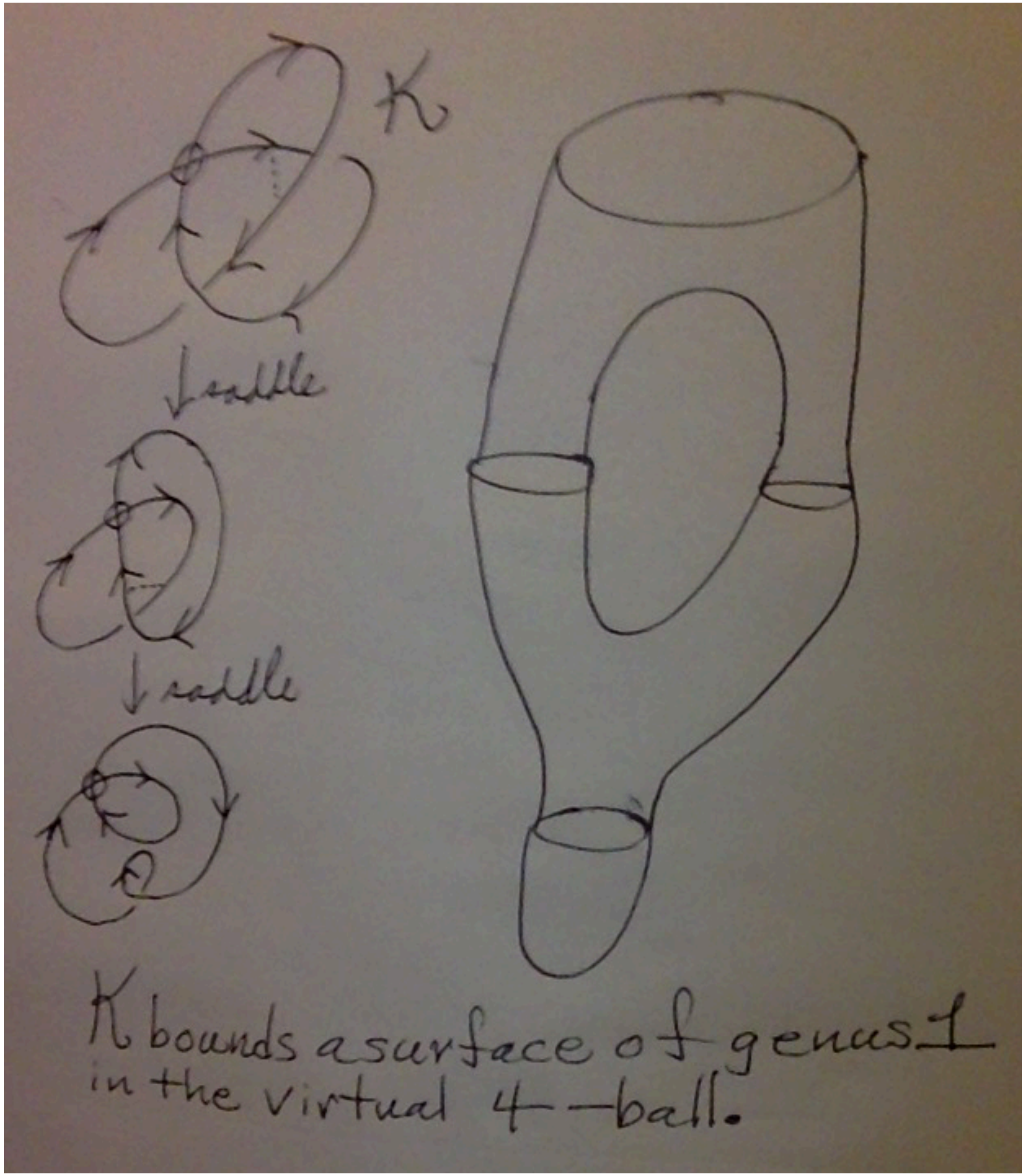


Seifert Circle(s) for K

Every virtual diagram K bounds a virtual orientable surface of genus $g = (1/2)(-r + n + 1)$ where r is the number of Seifert circles, and n is the number of classical crossings in K.
This virtual surface is the cobordism Seifert surface when K is classical.

$$r = 1, \quad n = 2, \quad g = (1/2)(-1 + 2 + 1) = 1$$

Figure 20: **Virtual Cobordism Seifert Surface**


K
saddle
saddle
K bounds a surface of genus 1
in the virtual 4-ball.

Heather Dye, Aaron Kaestner and LK, prove the following generalization of Rasmussen's Theorem, giving the four-ball genus of a positive virtual knot.

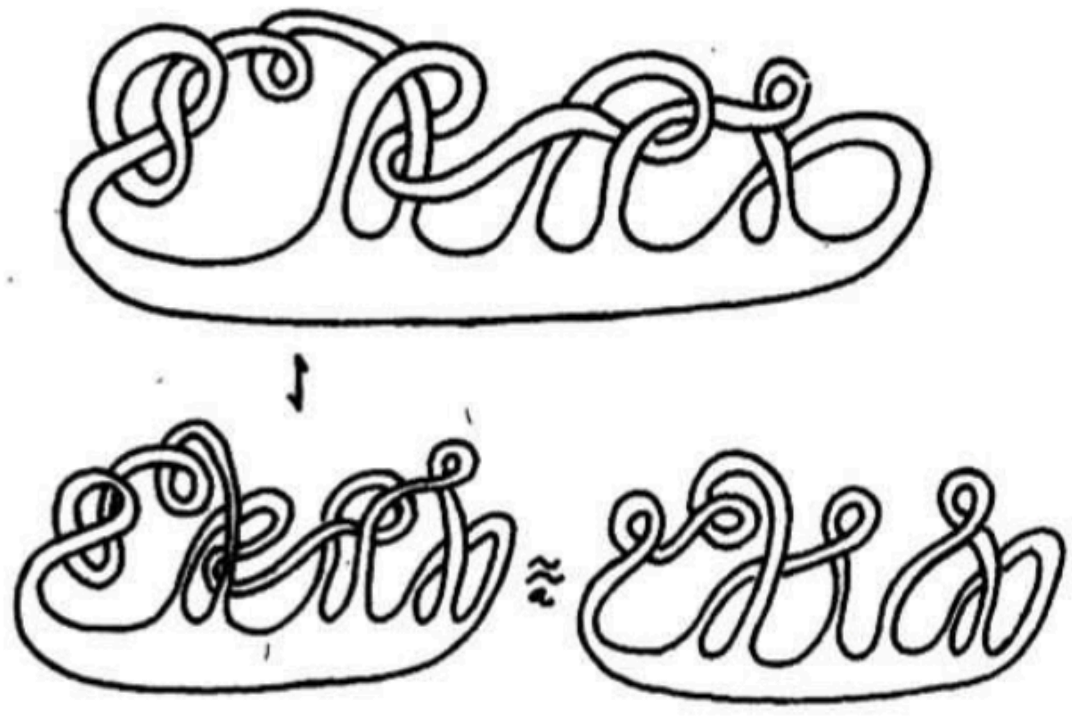
Theorem [2]. Let K be a positive virtual knot (all classcial crossings in K are positive), then the four-ball genus $g_4(K)$ is given by the formula

$$g_4(K) = (1/2)(-r + n + 1) = g(S(K))$$

where r is the number of virtual Seifert circles in the diagram K and n is the number of classcial crossings in this diagram. In other words, that virtual Seifert surface for K represents its minimal four-ball genus.

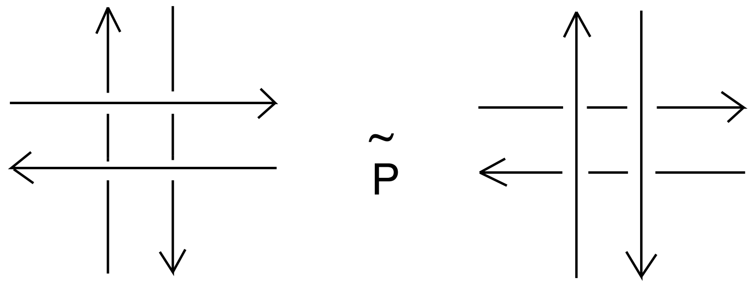
The virtual Seifert surface for positive virtual K represents the minimal four-ball genus of K.

The Theorem is proved by generalizing both Khovanov and Lee homology to virtual knots and generalizing the Rasmussen invariant to virtual knots.



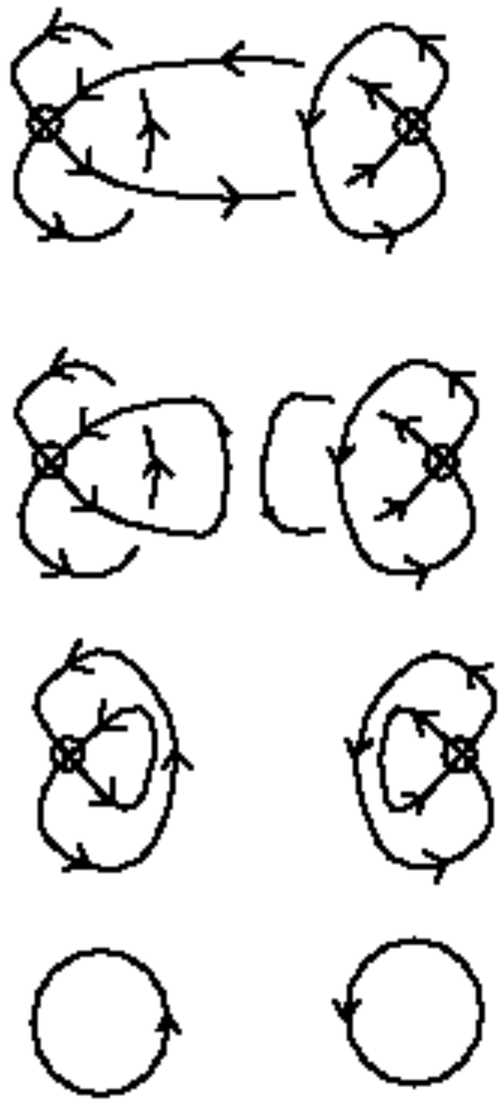
Classical Spanning Surfaces simplify by passing bands. Every classical knot is pass equivalent to either a trefoil or an unknot. Trefoil and unknot are distinguished by the Arf invariant.

Virtual Band Passing
VKT +



Classically there are two pass classes for knots: Trefoil and Unknot.

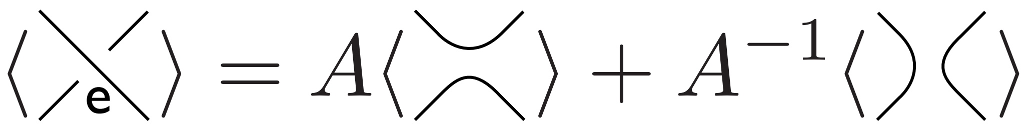
What are the pass classes for virtual knots and links?



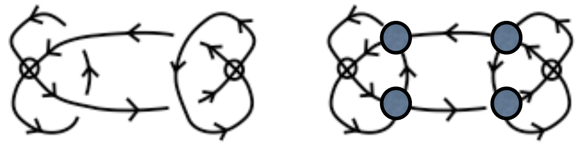
The Kishino
diagram gives a
virtual knot
that is slice but it
is not
PASS trivial.

Kishino is not pass trivial
since it is a non-trivial flat virtual knot. And its flat class IS its pass class since passing does not affect it as a flat.

Manturov Parity Bracket

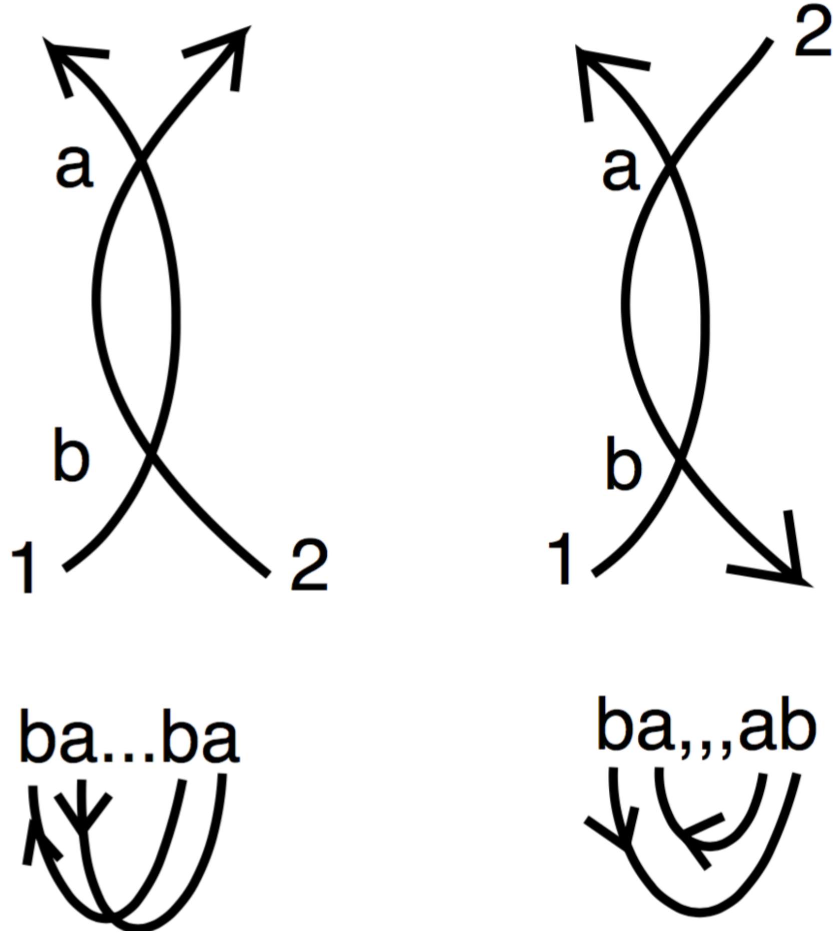


Kishino
Knot

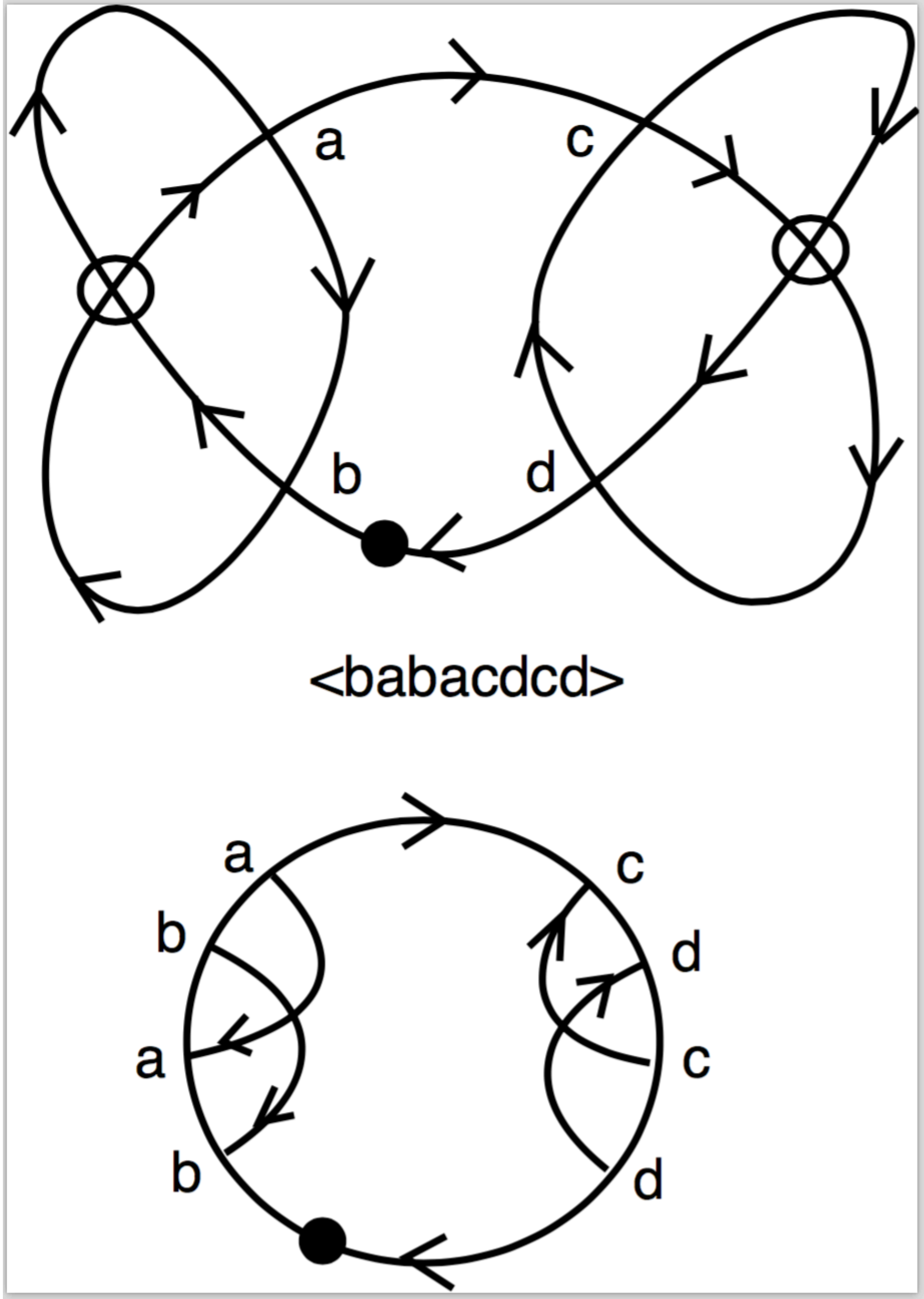


The Parity Bracket provides the simplest proof that the Kishino diagram is non-trivial.

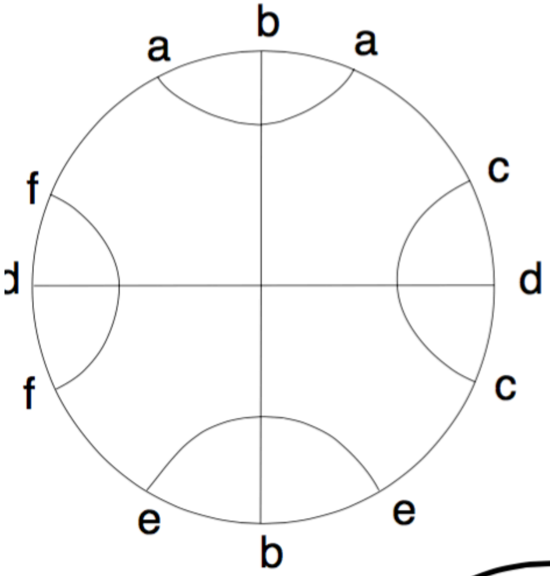
Parity bracket is calculated for virtuals and flat virtuals by replacing all odd crossings (odd interstice in Gauss code) with nodes. Then apply state sum with graphs (up to type two reducion) and polynomial coefficients. Kishino invariant is a single reduced diagram.



In flat Gauss code, two-moves require oppositely oriented parallel or crossed chords.

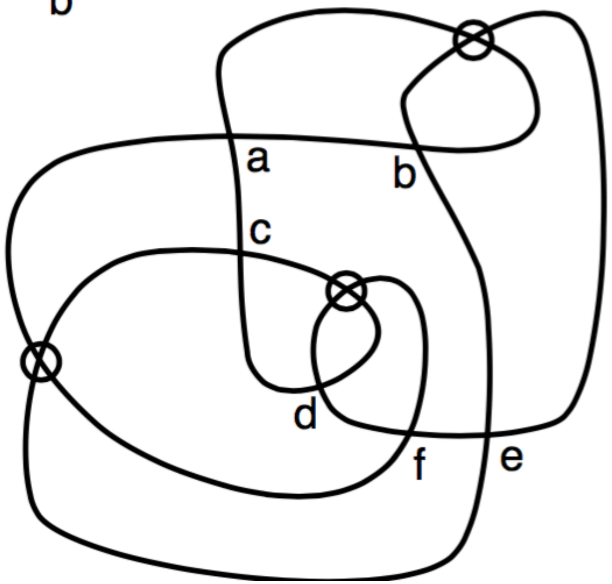


Reducing two-moves
are not available
on the flat
Kishino diagram.

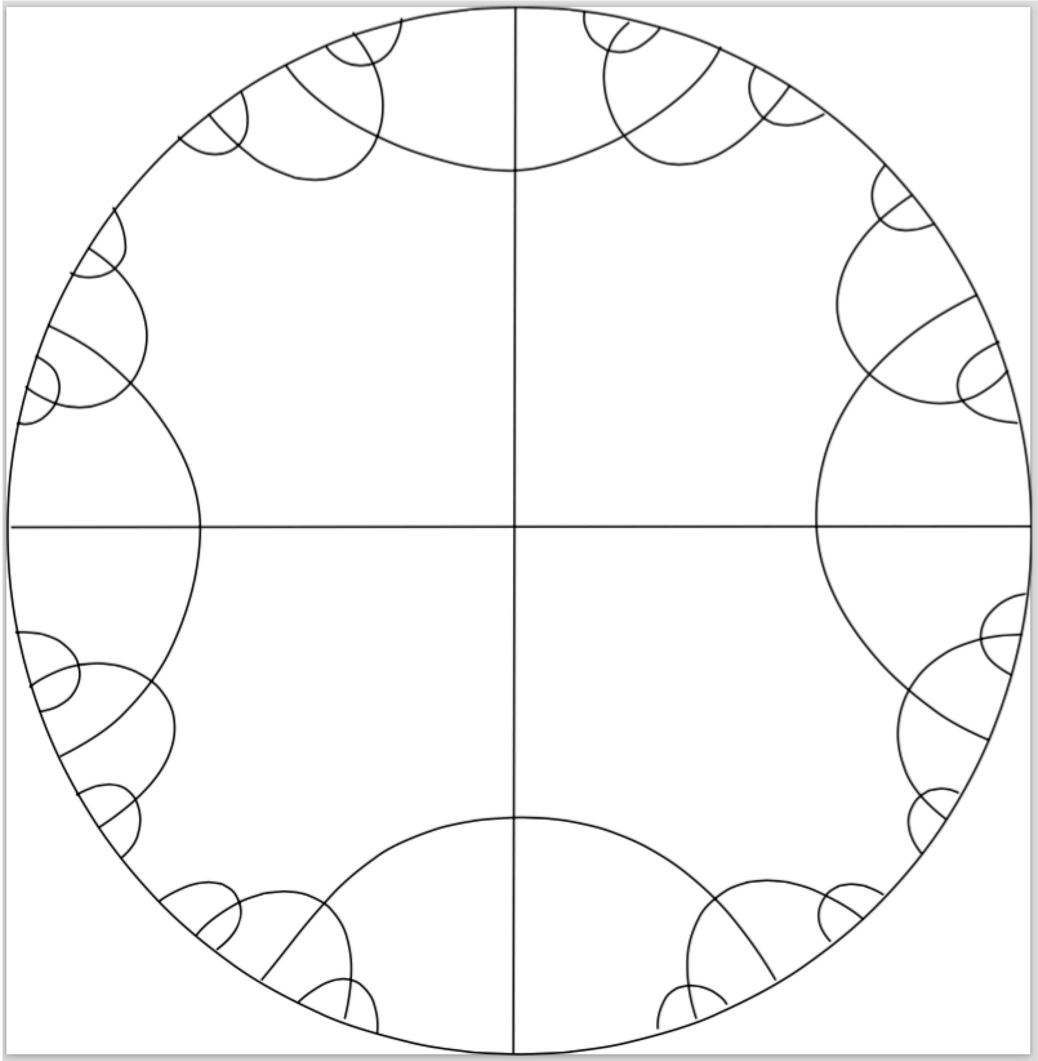


All odd crossings
and irreducible
as flat virtual diagram.

<abacdcebefdf>

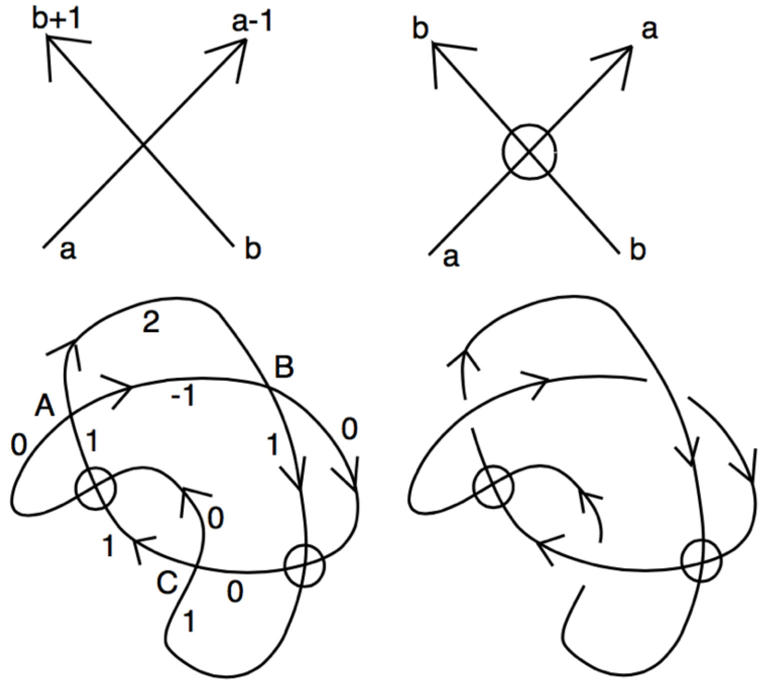


Here is another example of a flat with all odd crossings. It is non trivial by parity bracket and it is its own pass class.



This Gauss code schema shows how to produce infinitely many distinct flat virtuals, each their own pass class. Thus there are infinitely many distinct pass classes for virtual knots.

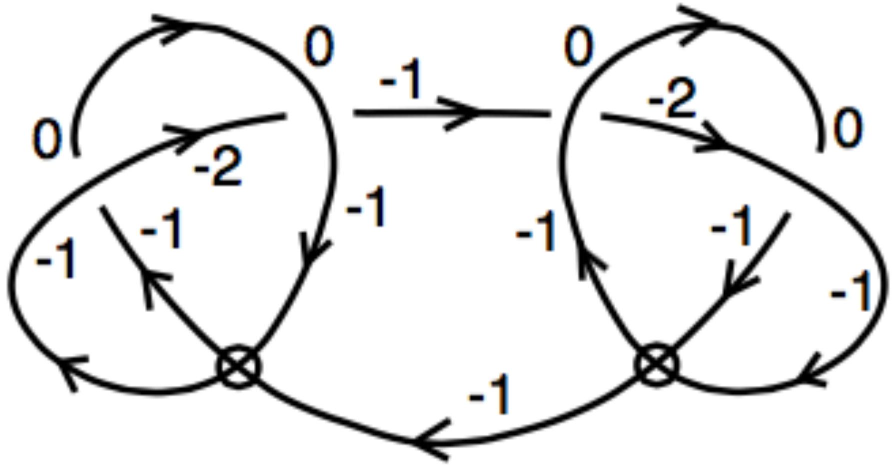
Affine Index Polynomial
(See LK and Folwazcny and variants from Henrich,Cheng,Dye,...)



	W_	W+
A	-2	+2
B	+2	-2
C	0	0

sgn(A) = sgn(B) = +1
sgn(C) = -1

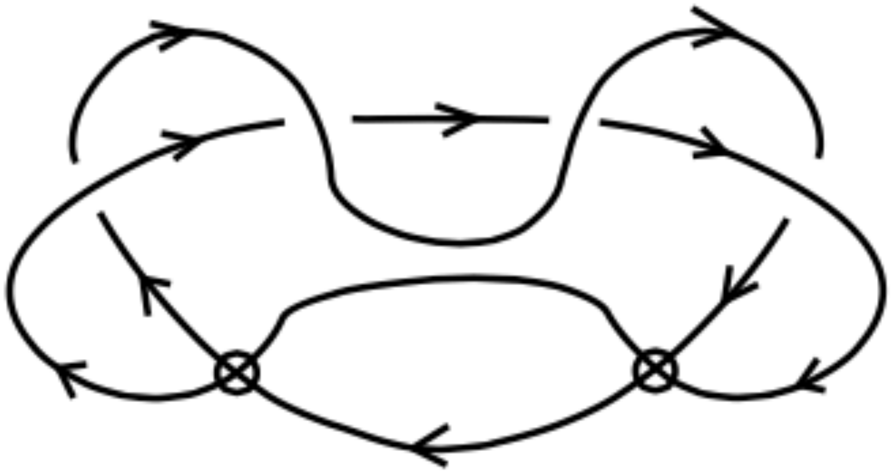
$P_K(t) = t^{-2} + t^{2} - 2$



K#K!

$$P_{K\#K!} = t^{-1} + t - t^{-1} - t = 0.$$

A slice knot with non-zero but cancelling weights.



$$P_K = \sum_c sgn(c)(t^{W_K(c)} - 1) = \sum_c sgn(c)t^{W_K(c)} - wr(K)$$

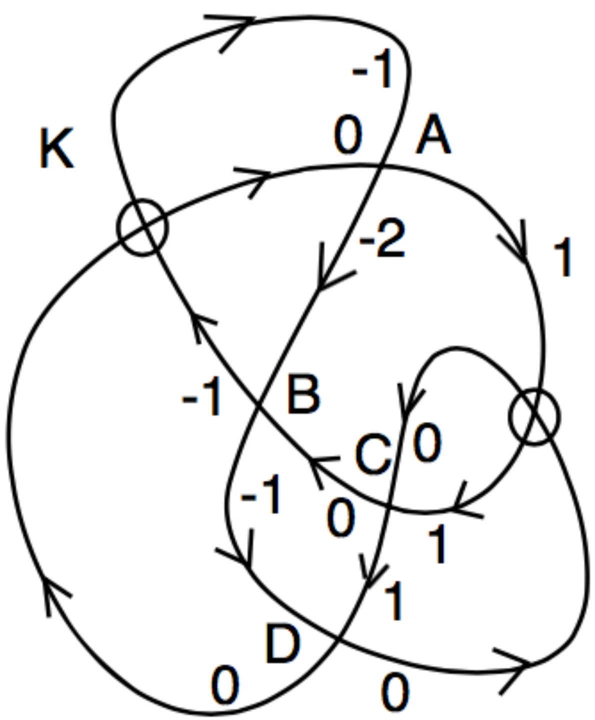
$$P_K = \sum_{n=1}^{\infty} wr_n(K)(t^n - 1)$$

$$wr_n(K) = \sum_{c:W_K(c)=n} sgn(c).$$

Remark. We define the *flat affine index polynomial*, PF_K, for a flat virtual knot K (in a flat virtual link the classical crossings are immersion crossings, neither over not under, Reidemeister moves are allowed independent of over and under, but virtual crossings still take detour precedence over classical crossings [14]) by the formula

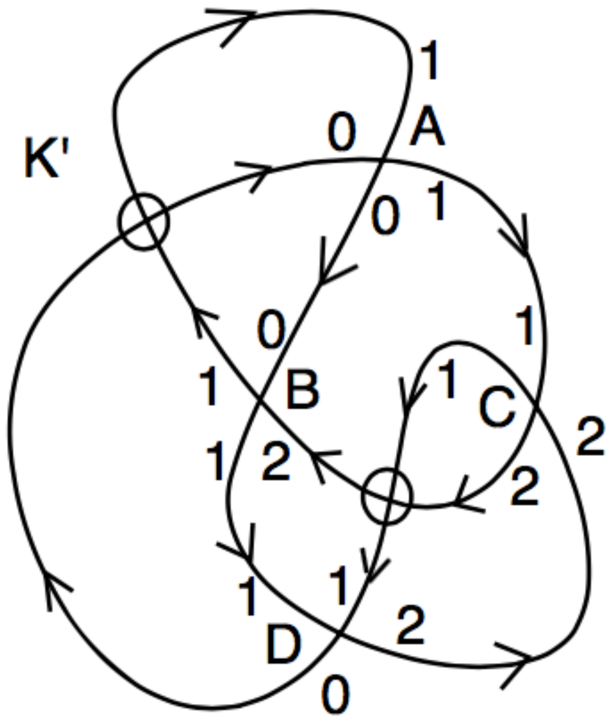
$$PF_K(t) = \sum_c (t^{|W_K(c)|} + 1)$$

where the polynomial is taken over the integers modulo two, but the exponents (the absolute values of the weights at the crossings) are integral. It is not hard to see that $PF_K(t)$ is an invariant of flat virtual knots, and that the concordance results of the present paper hold in the flat category for this invariant. These results will



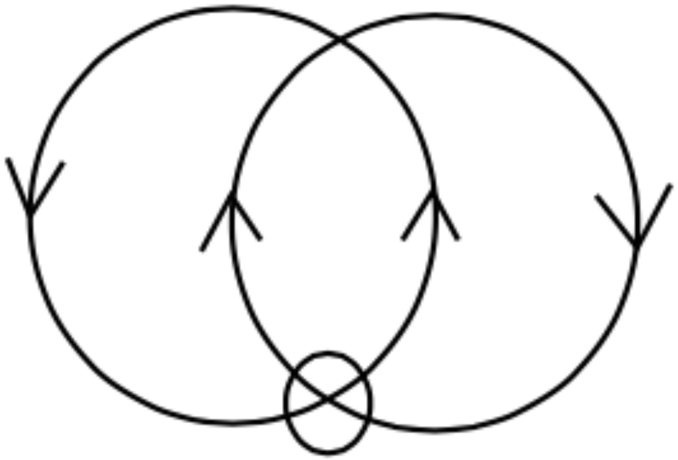
	w+	w-
A	-2	2
B	1	-1
C	0	0
D	1	-1

PF(K) = t^2 + 1 (mod 2)

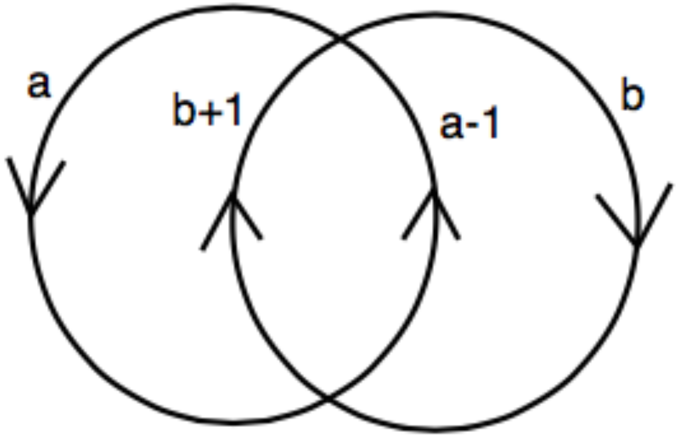


	w+	w-
A	0	0
B	1	-1
C	0	0
D	-1	1

PF(K') = 0 (mod 2)

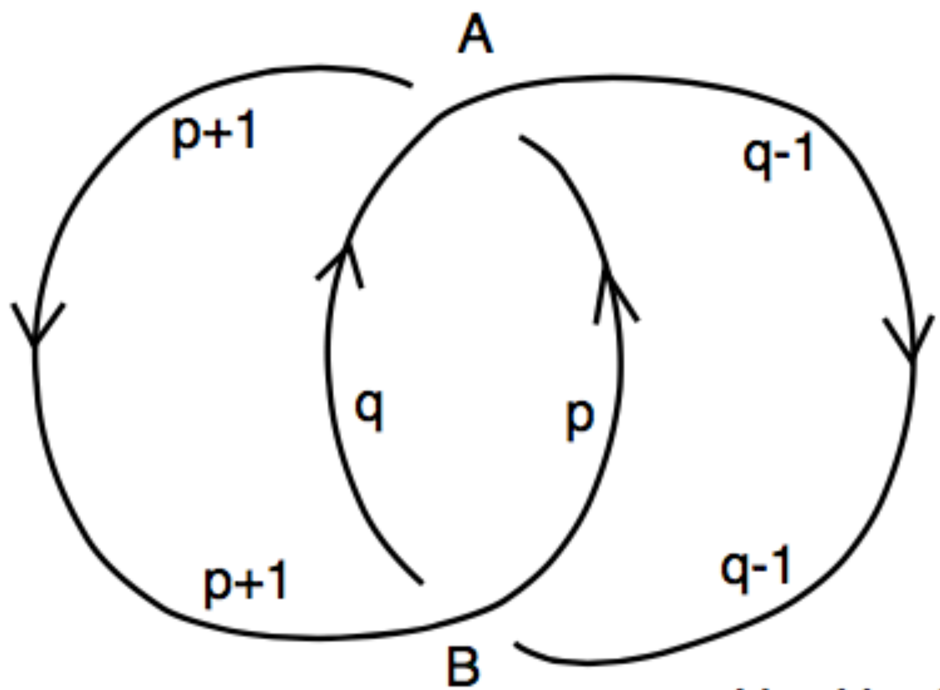


impossible to label



can be labeled

Index Invariant for Links



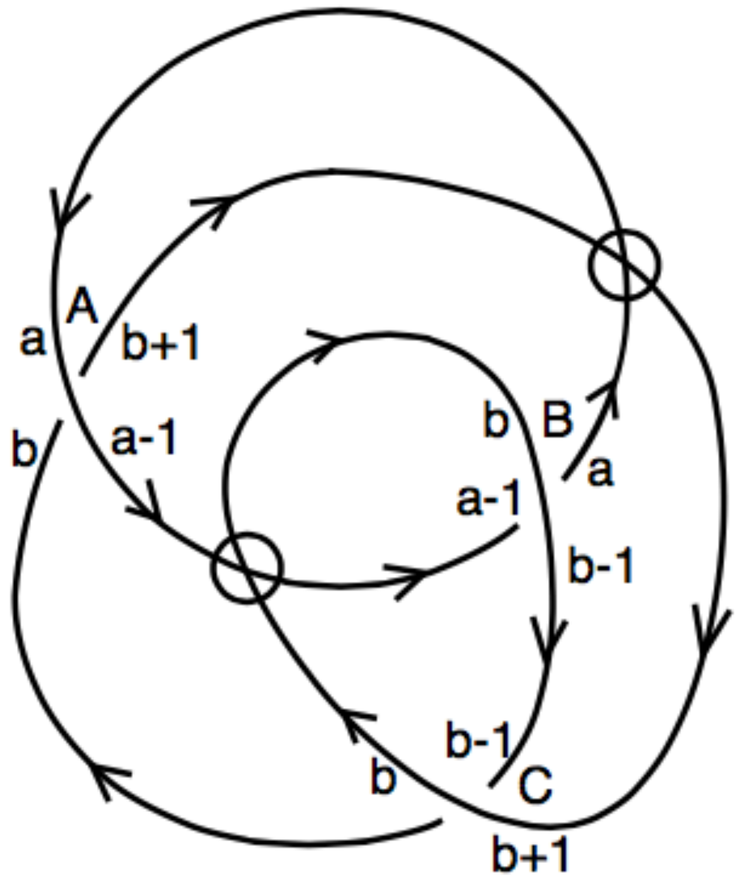
H = Hopf Link

$N = p - q$

$W(A) = q - p - 1 = -N - 1$

$W(B) = p - q + 1 = N + 1$

$$P_H(t) = t^{-N-1} + t^{N+1} - 2$$



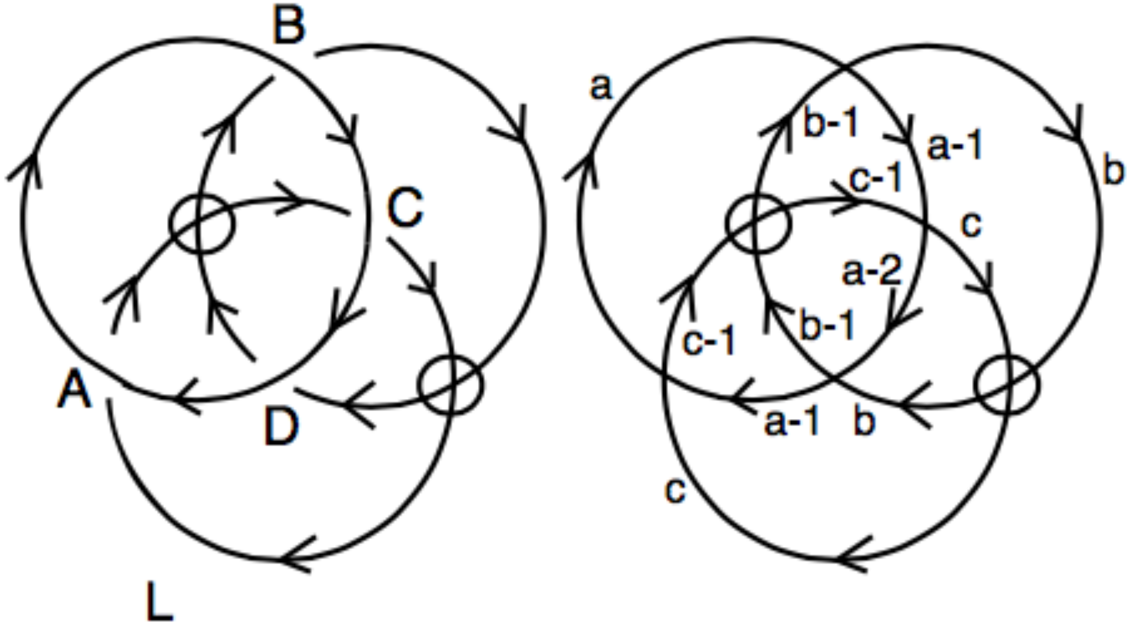
$N = a-b$

	w+	w-
A	N-1	1-N
B	-N	N
C	1	-1

Virtual Link L.

$PL = t^{N-1} + t^{-N} + t - 3$

Virtual Borromean Rings

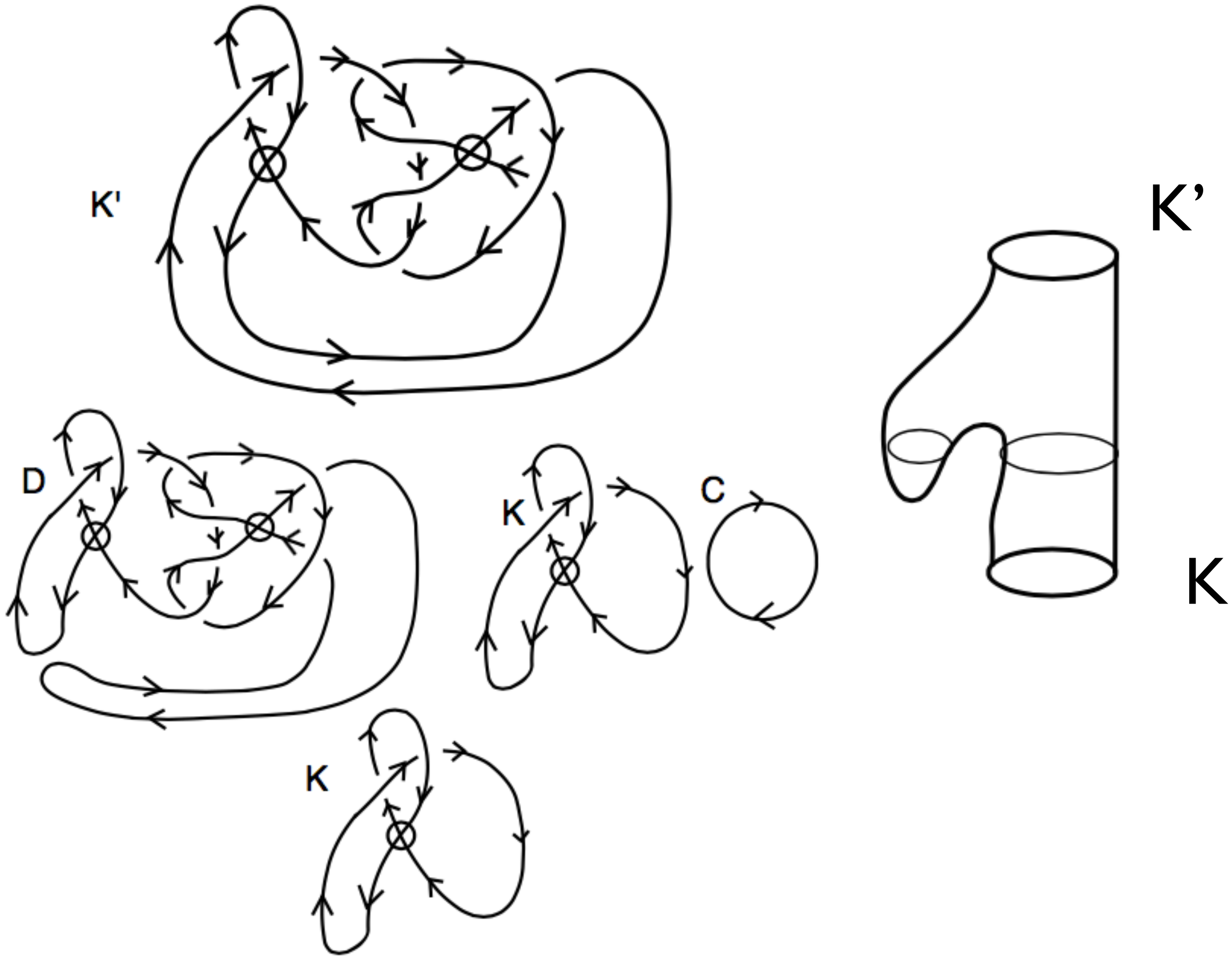


$$PL = -t^{M} + t^{N} + t^{M-1} - t^{N-1}$$

$N = a-b$, $M = a-c$

	w+	w-
A	-M	M
B	N	-N
C	M-1	-M+1
D	-N+1	N-1

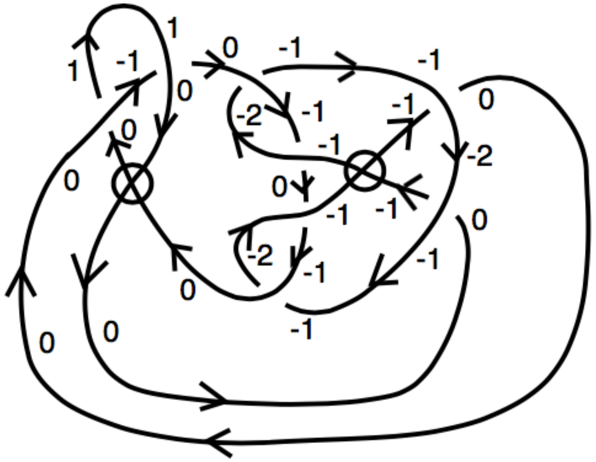
Concordances are Composed of Elementary Concordances
(Cancellation of Saddle and Max or Min)



Theorem. P_K is a concordance invariant.

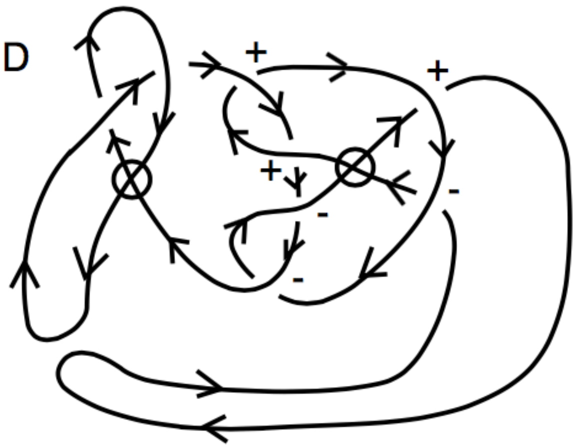
Proof. Concordances are compositions of elementary concordances.//

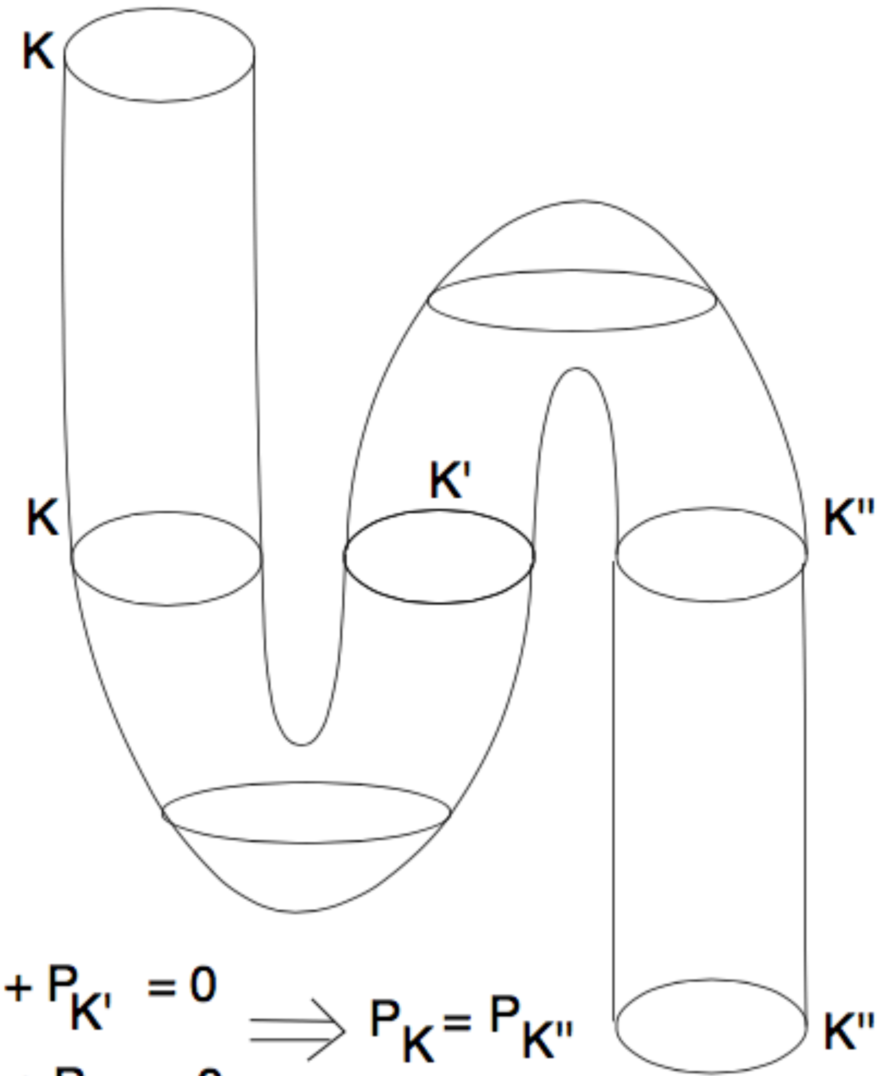
K'



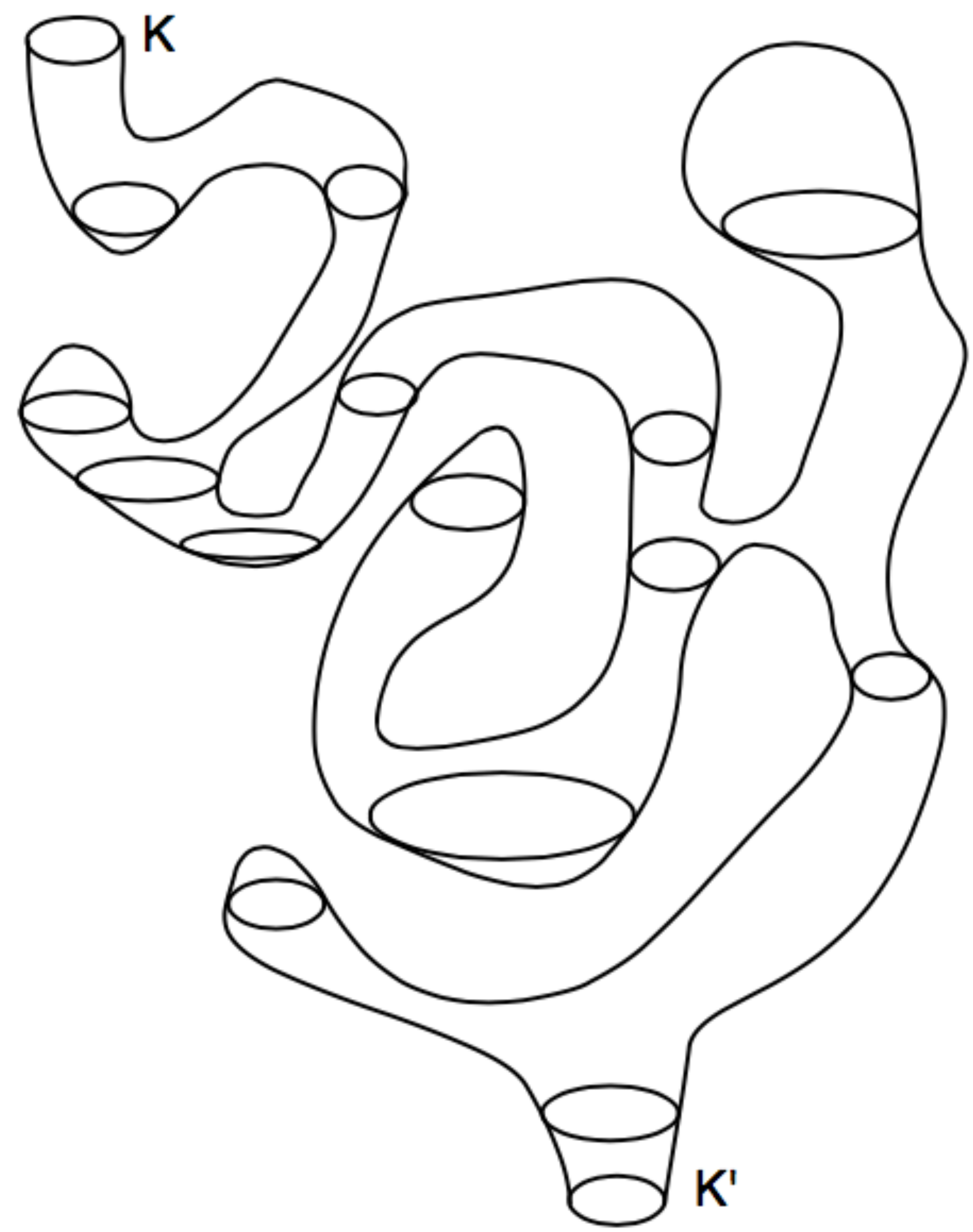
$PK' = t^{-1} + t + 2$

D





$$P_K + P_{K'} = 0$$
$$P_{K'} + P_{K''} = 0$$
$$\Longrightarrow P_K = P_{K''}$$

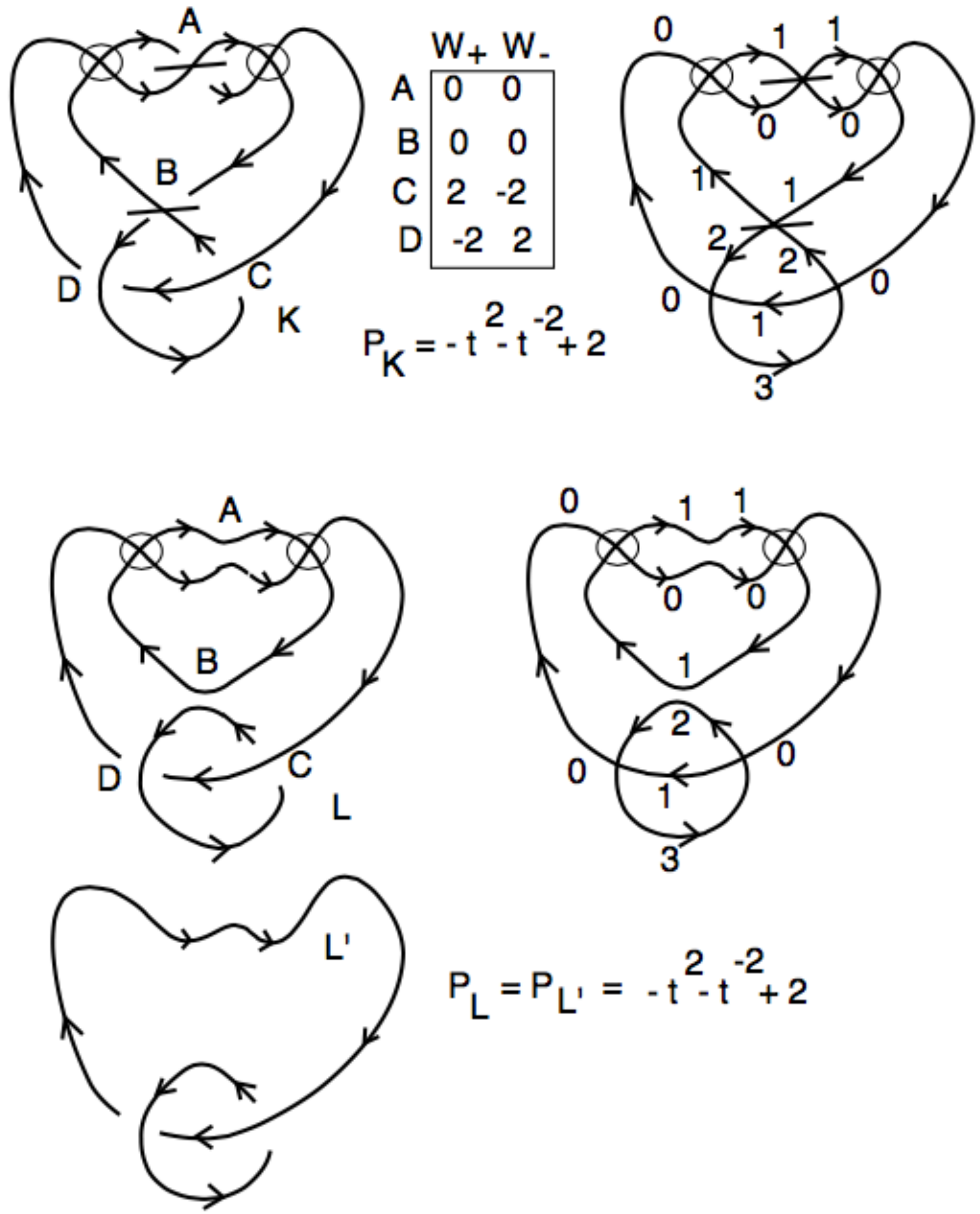

K
K'

Theorem. P_K is a concordance invariant.

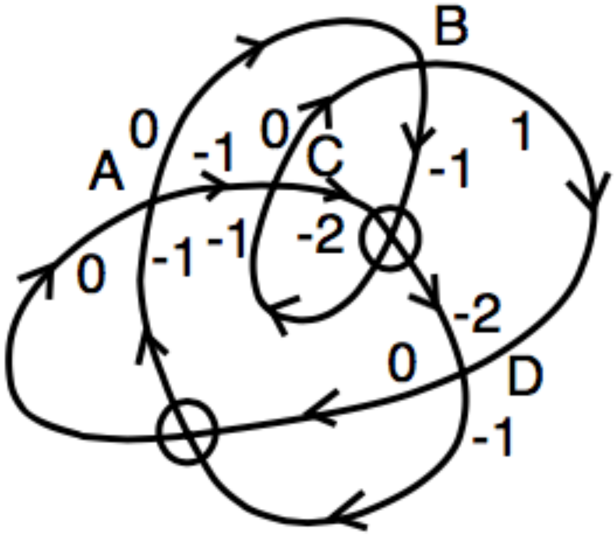
Proof. Concordances are compositions of elementary concordances.//

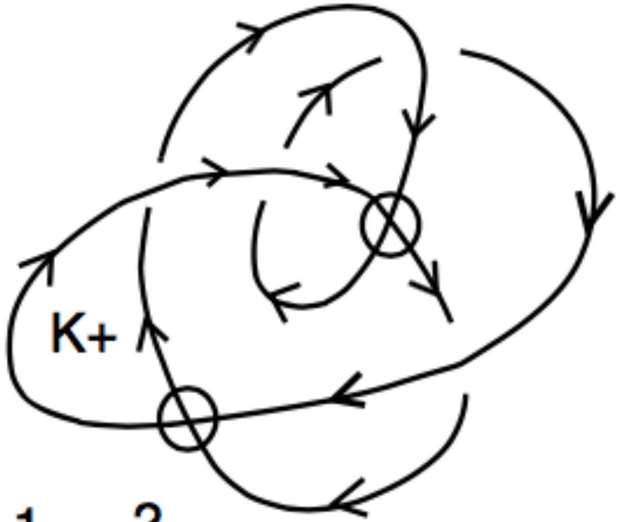
A special concordance of links is DEFINED to be a composition of elementary concordances.

P_K is an invariant of special concordance for links that have an affine labeling.



A labeled cobordism of a knot to a link.



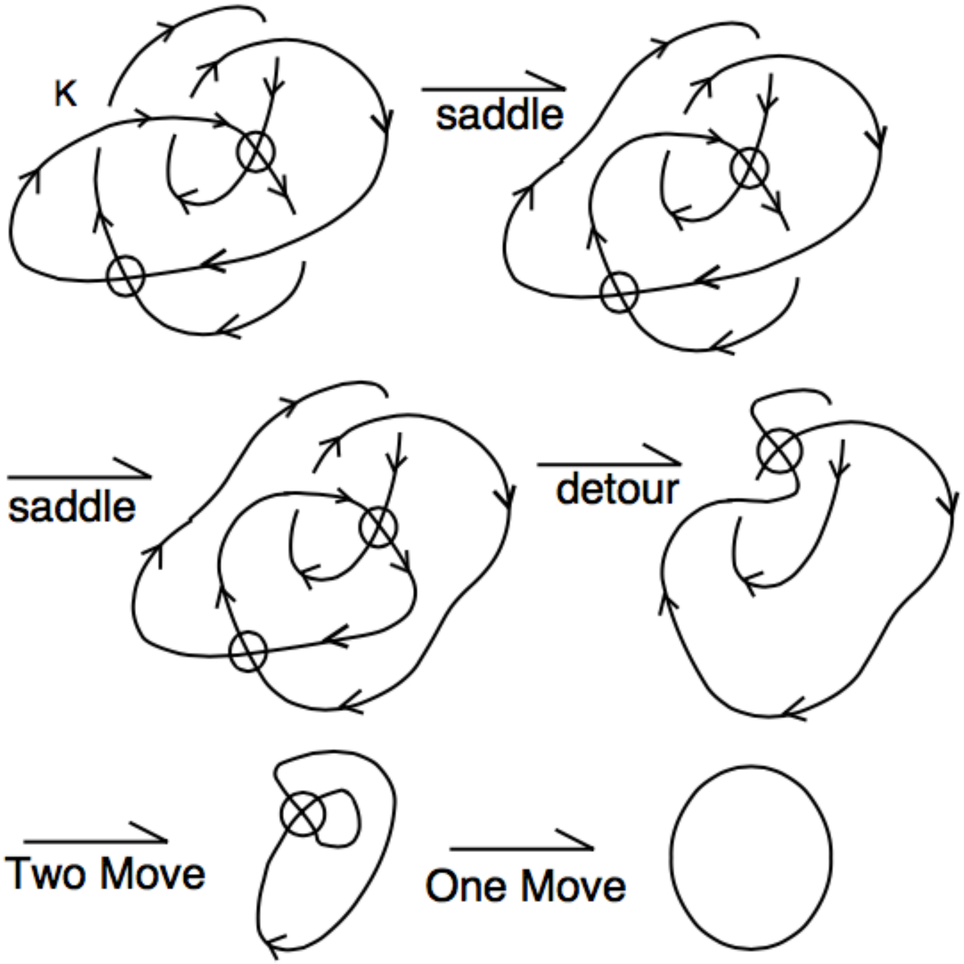


$PK+ = 2t^{-1} + t^{2} - 3$

	w+	w-
A	0	0
B	-1	1
C	-1	1
D	2	-2

$PK = t^{2} - 1$

$PK = t^{2} + t - t^{-1} - 1$



K bounds a virtual surface of genus one.

Hence, via P_K, K has genus one.

Thank you for your attention!