Topological Protection of Majorana fermion Qubits

Rukhsan ul haq

Postdoctoral Fellow,
Department of Physics, Zhejiang University
Hangzhou, China.

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Collaborator(s)

Outline of the talk

- Introduction
- Quantum Ising Model and duality
- Kitaev Chain and its Majorana edge modes
- Fermionic mode operators and spectrum doubling
- How to protect Majorana Fermion qubits?
- Majorana Fermions and Braiding
- Kitaev chain model and Yang-Baxter Equation
- Conclusions

Dirac Fermions and Second Quantization

$$\{c_i, c_i^\dagger\} = \delta_{ij} \quad c_i^2 = (c_i^\dagger)^2 = 0 \quad N = c^\dagger c \quad N^2 = N \tag{1}$$

where $c^\dagger$,c and N are creation,annihilation and number operator for a fermion.

$$\mid 1\rangle = c^\dagger \mid 0\rangle \quad \mid 0\rangle = c \mid 1\rangle \tag{2}$$

$$c \mid 0\rangle = c^\dagger \mid 1\rangle = 0 \tag{3}$$

Fermions have a vacuum state. Creation and annihilation operator are used to construct the states of fermions. Fermions have $U(1)$ symmetry, and hence number of fermions is conserved, and occupation number is a well-defined quantum number. Number of fermions in a state is given by the eigenvalue of number operator. Number operator is idempotent, and hence there are only two eigenvalues:$0, 1$. Also, different fermion operators anti-commute with each other and hence obey Fermi-Dirac statistics.

Algebra of Majorana Fermions

$$c = \frac{\gamma_1 + i\gamma_2}{\sqrt{2}} \quad c^\dagger = \frac{\gamma_1 - i\gamma_2}{\sqrt{2}} \tag{4}$$

Majorana Fermions obey Clifford Algebra $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$

$$\gamma_1 = \frac{c + c^\dagger}{\sqrt{2}} \quad \gamma_2 = \frac{i(c^\dagger - c)}{\sqrt{2}} \tag{5}$$

Majorana Fermions are their own anti-particles:$\gamma = \gamma^\dagger$.
Majorana Fermions do not satisfy Pauli Exclusion Principle.
There is no well-defined number operator for Majorana Fermions.
Majorana Fermions have Z_2 symmery and parity is the only good quantum number they have.

Transverse Field Ising Model

$$H = -J\sum_{i=1}^{N-1}\sigma_i^x\sigma_{i+1}^x - h_z\sum_{i=1}^{N}\sigma_i^z \quad (6)$$

This model has Z_2 symmetry due to which the global symmetry operator commutes with Hamiltonian.

$$\left[\prod_i \sigma_i^z, H\right] = 0 \quad (7)$$

Jordan-wigner Transformation maps spin operators into fermion operators.

$$c_i = \sigma_i^\dagger(\prod_{j=1}^{i-1}\sigma_i^z) \quad c_i^\dagger = \sigma_i^-((\prod_{j=1}^{i-1}\sigma_i^z) \quad (8)$$

$$H = -J\sum_{i=0}^{N-1}(c_i^\dagger c_{i+1} + h.c.) - J\sum_{i=0}^{N-1}c_i^\dagger c_{i+1}^\dagger + h.c. - 2h\sum_{i=0}^{N}c_i^\dagger c_i$$

Kitaev Chain and Majorana Edge Modes

Kitaev introduced p-wave chain model:

$$H = -t\sum_{i=0}^{N-1}(c_i^\dagger c_{i+1} + h.c.) + \triangle\sum_{i=0}^{N-1} c_i^\dagger c_{i+1}^\dagger + h.c. - \mu\sum_{i=0}^{N} c_i^\dagger c_i$$

Using Majorana representation of fermions:

$$c_i = \frac{\gamma_{1,i} - i\gamma_{2,i}}{\sqrt{2}} \quad c_i^\dagger = \frac{\gamma_{1,i} + i\gamma_{2,i}}{\sqrt{2}} \tag{9}$$

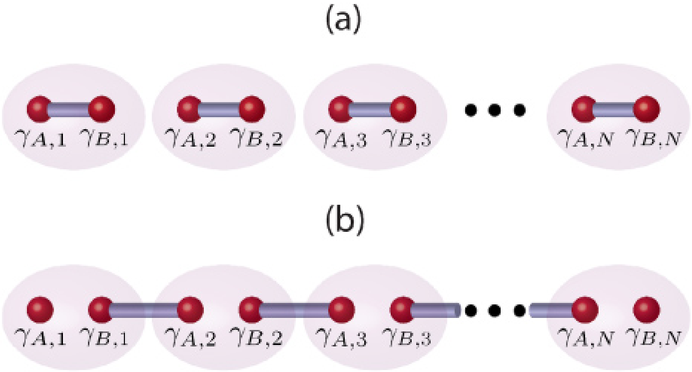
$$H = it\sum_{i=0}^{N-1}(\gamma_{1,i}\gamma_{2,i+1} - \gamma_{2,i}\gamma_{1,i+1}) + i\Delta\sum_{i=0}^{N-1}(\gamma_{1,i}\gamma_{2,i+1} + \gamma_{2,i}\gamma_{1,i+1}) \tag{10}$$

$$-\mu\sum_{i=0}^{N}(\frac{1}{2} - i\gamma_{1,i}\gamma_{2,i}) \tag{11}$$

Due to the superconducting term,there is no number conservation,only parity is conserved.

$$P = i\gamma_1\gamma_2 = 1 - 2c^\dagger c \tag{12}$$

Majorana Edge Modes in Kitaev Chain



Ref:Jason Alicea Rep. Prog. Phys.75 (2012)

Topological Phase of Kitaev Chain

Choosing $\mu = 0$ and $t = \Delta$ the Hamiltonian becomes.

$$H = 2it \sum_{i=0}^{N-1} \gamma_{1,i}\gamma_{2,i+1} \tag{13}$$

We can define a complex fermion:

$$a_i = \frac{\gamma_{2,i+1} - i\gamma_{1,i}}{\sqrt{2}} \tag{14}$$

The Hamiltonian becomes:

$$H = \left(t \sum_{i=0}^{N-1} a_i^\dagger a_i - \frac{1}{2} \right) \tag{15}$$

$$a_0 = \frac{\gamma_{1,N} - i\gamma_{2,0}}{\sqrt{2}} \quad H_b = \epsilon_0 a_0^\dagger a_0 \quad \epsilon_0 = 0 \tag{16}$$

The Hamiltonian has double degeneracy which is protected by parity symmetry and hence this is topological degeneracy.

Topological Order and Majorana fermions

- Majorana fermions(actually **Majorana Zero Modes**) have attracted lot of attention in condensed matter physics community.
- Majorana fermions are the promising candidates for topological quantum computing because of their non-abelian anyonic statistics.
- Majorana fermions occur in quantum Hall fluids, topological superconductors, quantum spin liquids, Multi-channel Kondo models.
- Existence of Majorana fermions is signature of topological order.
- Kitaev chain model can be obtained from Transverse field Ising model(TFIM).
- Why there is topological order in Kitaev chain and not in TFIM?
- Some attempts to answer this question: Greiter *et al*,Cobanerra *et al*
- However they have just explored the duality between the models and not explained the emergence of topological order in Kitaev chain.

Ref: Annals of Physics, **351**,1026(2014), Phys. Rev. B.**87**, 0411705(2013)

Algebra of Majorana Doubling

- Kitaev found Majorana edge modes in his model.
- Each state in Kitaev chain spectrum has degenerate partner due to parity symmetry like as time reversal symmetry leads to Kramers pairs.
- Lee and Wilzeck(PRL **111**(2013)) showed that in Kitaev chain there are more symmetries which lead to doubled spectrum.

$$\{1, \gamma_1 = a_1, \gamma_2 = a_2, \gamma_3 = a_3, \gamma_{12} = a_1 a_2, \gamma_{23} = a_2 a_3, \gamma_{31} = a_3 a_1, \gamma_{123} = a_1 a_2 a_3\}$$

Hamiltonian for three Majorana fermions:

$$H_m = -i(\alpha b_1 b_2 + \beta b_2 b_3 + \gamma b_3 b_1)$$

Naively one would take it for spin Hamiltonian but there are subtle differences:

$$\Gamma \equiv -i b_1 b_2 b_3$$
$$\Gamma^2 = 1 \quad [\Gamma, b_j] = 0 \quad [\Gamma, H_m] = 0 \quad \{\Gamma, P\} = 0$$

$\{\Gamma, P\} = 0$ leads to the even-odd pair for each energy value.

Topological Order and Majorana Mode Operators

- **Fermionic zero modes are one of the very important signatures of topological order.**
- **Fermionic mode operators give a neat way to find topological order in a given Hamiltonian.**

A fermionic zero mode is an operator Γ such that

- Commutes with Hamiltonian:$[H, \Gamma] = 0$
- anticommutes with parity:$\{P, \Gamma\} = 0$
- has finite "normalization" even in the $L \to \infty$ limit:$\Gamma^\dagger \Gamma = 1$.

We find that the same Majorana mode operator which leads to the spectrum doubling also leads to the topological order in Kitaev chain model.
Majorana mode operator is not present for the spin Hamiltonian and hence there is no topological order over there.

Topological protection and quantum operator algebra

Topological degeneracy and topological protection can be understood in a more general way based on the operator algebra of symmetry generators.

$$[P, H] = [Q, H] = 0 \quad \{P, Q\} = 0 \quad P^2 = Q^2 = 0 \tag{17}$$

P and Q are symmetry operators of the Hamiltonian H which anti-commute with each other.
Because P and Q commute with H, so they will have same eigenstates but because P and Q anti-commute, so the eigenvalues can not be same.

$$P \mid \Psi\rangle = m \mid \Psi\rangle \quad Q(P \mid \Psi\rangle) = mQ(\mid \Psi\rangle) \quad P(Q \mid \Psi\rangle) = -m(Q \mid \Psi\rangle) \tag{18}$$

For every state with eigenvalue m, there is another state with eigenvalue -m: Doubling of the spectrum.

Topological protection for Majorana fermion chains

We consider a system which has $2N+1$ Majorana fermions. These Majorana fermions will span a vector space of dimensionality 2^{2N+1} corresponding to the number of linearly independent generators of Clifford algebra. These generators can be written as

$$
\begin{aligned}
& 1, \gamma_1, \gamma_2 \dots, \gamma_{2N+1}, \\
& \gamma_1\gamma_2, \gamma_1\gamma_3 \dots. \quad (19) \\
& \gamma_1\gamma_2\gamma_3 \dots. \quad (20) \\
& \vdots \cdots \ \gamma_1\gamma_2 \dots . \gamma_{2N+1} \quad (21)
\end{aligned}
$$

The most general local quadratic Hamiltonian for the Majorana fermions can be written as

$$H = i \sum_{ij} h_{ij}\gamma_i\gamma_j \quad (22)$$

Due to the anti-commuting nature of the Majorana fermions, $h_{ij} = -h_{ji}$. This Hamiltonian has manifest $Z2$ symmetry and consequently the Hamiltonian can be diagonalized in the parity eigenbasis.

Topological Degenercay (continued)

Now we the generalized τ operator which commutes with Hamiltonian.

$$\tau = i^{\frac{2N+1}{N}} \gamma_1 \gamma_2 \ldots \gamma_{2N+1} \tag{23}$$

This operator is not only the symmetry operator of the Hamiltonian but it also squares to unity and anti-commutes with parity and hence is emergent Majorana mode operator.
Now we can see that this odd Majorana fermion chain has the algebraic structure needed for topological degeneracy: P is the parity operator and τ is Q operator.

$$[P, H] = [\tau, H] = 0 \quad \{P, \tau\} = 0 \quad P^2 = \tau^2 = 1 \tag{24}$$

So, we show that chain of odd number of Majorana fermions will have topological degeneracy.

Topological Degeneracy and Braid Group

Braid group generators satisfy:

$$T_i T_j = T_j T_i \quad | i - j |> 1$$
$$T_i T_j T_i = T_j T_i T_j \quad | i - j |= 1 \tag{25}$$

Ivanov showed that Majorana fermions give representation of braid group.

$$\tau_i = exp(\frac{\pi}{4}\gamma_{i+1}\gamma_i) = \frac{1}{\sqrt{2}}(1 + \gamma_{i+1}\gamma_i) \tag{26}$$

Braiding of Majorana fermions happens only in topological phase.

$$H = 2it \sum_{i=0}^{N-1} \gamma_{1,i}\gamma_{2,i+1} \tag{27}$$

Ref:Ivanov,PRL, **86** (2001) Kauffman-Lomanaco, NJP **4**(2002)
Kauffman-Lomanaco,arxiv:1603.07827

Ivanov Represenation

Braiding operators arise from a row of Majorana Fermions $\{\gamma_1, \cdots \gamma_n\}$ as follows: Let

$$\sigma_i = (1/\sqrt{2})(1 + \gamma_{i+1}\gamma_i).$$

Note that if we define

$$\lambda_k = \gamma_{i+1}\gamma_i$$

for $i = 1, \cdots n$ with $\gamma_{n+1} = \gamma_1$, then

$$\lambda_i^2 = -1$$

and

$$\lambda_i\lambda_j + \lambda_j\lambda_i = 0$$

where $i \neq j$. From this it is easy to see that

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

for all i and that $\sigma_i\sigma j = \sigma j\sigma_i$ when $|i - j| > 2$. Thus we have constructed a representation of the Artin braid group from a row of Majorana fermions. This construction is due to Ivanov and he notes that

$$\sigma_i = e^{(\pi/4)\gamma_{i+1}\gamma_i}.$$

Type II Braid Group Representation

$$M_i M_{i\pm1} = -M_{i\pm1} M_i, \quad M^2 = -I, \tag{28}$$

$$M_i M_j = M_j M_{i,} \quad \left| i - j \right| \geq 2. \tag{29}$$

The operators M_i take the place here of the products of Majorana Fermions $\gamma_{i+1}\gamma_i$ in the Ivanov picture of braid group representation in the form

$$\sigma_i = (1/\sqrt{2})(1 + \gamma_{i+1}\gamma_i).$$

This goes beyond the work of Ivanov, who examines the representation on Majoranas obtained by conjugating by these operators. The Ivanov representation is of order two, while this representation is of order eight. The Bell-Basis Matrix B_{II} is given as follows:

$$B_{II} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}(I + M) \quad \left(M^2 = -1\right) \tag{30}$$

Majorana Fermions and TLA

We define A and B as $A = \gamma_i \gamma_{i+1}$, $B = \gamma_{i-1}\gamma_i$ where $A^2 = B^2 = -1$.

$$U = (1 + iA) \quad V = (1 + iB), \tag{31}$$

$$U^2 = 2U \quad V^2 = 2V, \tag{32}$$

$$UVU = V \quad VUV = U, \tag{33}$$

Thus a Majorana fermion representation of TLA is given by:

$$U_k = \frac{1}{\sqrt{2}}(1 + i\gamma_{k+1}\gamma_k), \tag{34}$$

$$U_k^2 = \sqrt{2} U_k, \tag{35}$$

$$U_k U_{k\pm1} U_k = U_k, \tag{36}$$

$$U_k U_j = U_j U_k \quad for |k - j| \geq 2. \tag{37}$$

Hence we have a representation of the Temperley-Lieb algebra with loop value $\sqrt{2}$. Using this representation of the Temperley-Lieb algebra, we can construct Jones representation of the braid group.

Kitaev chain and Yang-Baxter Equation

$$\check{R}_i(\theta) = e^{\theta\gamma_{i+1}\gamma_i} \tag{38}$$

Then $\check{R}_i(\theta)$ satisfies the full Yang-Baxter equation with rapidity parameter θ. That is, we have the equation

$$\check{R}_i(\theta_1)\check{R}_{i+1}(\theta_2)\check{R}_i(\theta_3) = \check{R}_{i+1}(\theta_3)\check{R}_i(\theta_2)\check{R}_{i+1}(\theta_1) \tag{39}$$

We can construct a Kitaev chain based on the solution $\check{R}_i(\theta)$ of the Yang-Baxter Equation. Let a unitary evolution be governed by $\check{R}_i(\theta)$. When θ in the unitary operator $\check{R}_i(\theta)$ is time-dependent, we define a state $|\psi(t)\rangle$ by $|\psi(t)\rangle = \check{R}_i|\psi(0)\rangle$. With the Schrödinger equation $i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$ one obtains:

$$i\hbar\frac{\partial}{\partial t}[\check{R}_i|\psi(0)\rangle] = \hat{H}(t)\check{R}_i|\psi(0)\rangle. \tag{40}$$

From YBE to Hamiltonian

We can construct a Kitaev chain based on the solution $\check{R}_i(\theta)$ of the Yang-Baxter Equation. Let a unitary evolution be governed by $\check{R}_i(\theta)$. When θ in the unitary operator $\check{R}_i(\theta)$ is time-dependent, we define a state $|\psi(t)\rangle$ by $|\psi(t)\rangle = \check{R}_i|\psi(0)\rangle$. With the Schrödinger equation $i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$ one obtains:

$$i\hbar\frac{\partial}{\partial t}[\check{R}_i|\psi(0)\rangle] = \hat{H}(t)\check{R}_i|\psi(0)\rangle. \tag{41}$$

Then the Hamiltonian $\hat{H}_i(t)$ related to the unitary operator $\check{R}_i(\theta)$ is obtained by the formula:

$$\hat{H}_i(t) = \mathrm{i}\hbar\frac{\partial\check{R}_i}{\partial t}\check{R}_i^{-1}. \tag{42}$$

Substituting $\check{R}_i(\theta) = \exp(\theta\gamma_{i+1}\gamma_i)$ into equation (42), we have:

$$\hat{H}_i(t) = \mathrm{i}\hbar\dot{\theta}\gamma_{i+1}\gamma_i. \tag{43}$$

Two phases of Kitaev chain model

If we only consider the nearest-neighbour interactions between Majorana Fermions, and extend equation to an inhomogeneous chain with $2N$ sites, the derived model is expressed as:

$$\hat{H} = \mathrm{i}\hbar \sum_{k=1}^{N} (\dot{\theta}_1 \gamma_{2k}\gamma_{2k-1} + \dot{\theta}_2 \gamma_{2k+1}\gamma_{2k}), \tag{44}$$

with $\dot{\theta}_1$ and $\dot{\theta}_2$ describing odd-even and even-odd pairs, respectively. They then analyze the above chain model in two cases:

1. $\dot{\theta}_1 > 0$, $\dot{\theta}_2 = 0$.
2. $\dot{\theta}_1 = 0$, $\dot{\theta}_2 > 0$.

Thus the Hamiltonian derived from $\breve{R}_i(\theta(t))$ corresponding to the braiding of nearest Majorana fermion sites is exactly the same as the $1D$ wire proposed by Kitaev, and $\dot{\theta}_1 = \dot{\theta}_2$ corresponds to the phase transition point in the "superconducting" chain. By choosing different time-dependent parameter θ_1 and θ_2, one finds that the Hamiltonian $\hat{H}$ corresponds to different phases.

Conclusions

- Clifford algebra of Majorana Fermions leads to richer structure and larger group of emergent symmetries.
- There is doubling in the spectrum due to the Fermionic zero mode operators.
- The double degeneracy in the topological phase of Majorana fermion chain is topological and symmetry protected.
- Topological degeneracy can be understood in terms of two sets of the symmetry operators of the Hamiltonian which anti-commute among themselves.
- Majorana fermions provide a new type of the unitary representation of the braid group.
- Majorana fermions also provide representation of the TLA and extra-special group.
- Topological order in Majorana fermion systems is related to topological entanglement as given in Yang-Baxter equation.

Rukhsan Ul Haq and L.H. Kauffman,arxiv:1704.00252